Caricature of Hydrodynamics for Lattice Dynamics

T.V. Dudnikova [1]
Elektrostal Polytechnical Institute
Elektrostal 144000, Russia
e-mail: tdudnikov@mail.ru

Abstract

The lattice dynamics in $\mathbb{Z}^d$, $d\ge 1$, is considered. The initial data are supposed to be random function. We introduce the family of initial measures $\{\mu_0^\varepsilon,\varepsilon>0\}$ depending on a small scaling parameter ε. We assume that the measures μ_0^ε are locally homogeneous for space translations of order much less than ε^{-1} and nonhomogeneous for translations of order ε^{-1}. Moreover, the covariance of μ_0^ε decreases with distance uniformly in ε. Given $\tau\in\mathbb{R}\setminus 0$, $r\in\mathbb{R}^d$, and $\kappa>0$, we consider the distributions of random solution in the time moments $t=\tau/\varepsilon^\kappa$ and at lattice points close to $[r/\varepsilon]\in\mathbb{Z}^d$. The main goil is to study the asymptotics of these distributions as $\varepsilon\to 0$ and derive the limit hydrodynamic equations of the Euler or Navier-Stokes type. The similar results are obtained for lattice dynamics in the half-space $\mathbb{Z}^d_+$.

Key words and phrases: harmonic crystals, random initial data, covariance matrices, weak convergence of measures, Gaussian measures, hydrodynamic limit, hydrodynamic space-time scaling, energy transport equation, Euler and Navier–Stokes equations

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1 Introduction

One of the central problems in nonequilibrium statistical physics is the derivation of hydrodynamic equations of fluid from the microscopic Hamilton dynamics. The main fluid equations are the Euler and Navier–Stokes equations. The idea that the Euler equation of fluid dynamics could be derived from microscopic dynamics goes back to Morrey [22]. A systematic explanation of some basic ideas and first results are presented in the survey papers by De Masi, Ianiro, Pellegrinotti, and Presutti [3], by Dobrushin, Sinai and Sukhov [5], and by Spohn [23].

One approach is to derive the Euler and Navier–Stokes equations from the Boltzmann equation (see, for example, [4]). However, the Boltzmann equation is not a microscopic model, it should itself be derived as a scaling limit of a more basic model. An alternate approach is to study the hydrodynamic behaviour of some simplified or *idealized* models of interacting particles. For models of *stochastic* dynamics, the results were obtained by Yau *et al.* (see, for example, [18, 20, 21], the survey paper by Fritz [19] and the bibliography there). *Deterministic* models, where only initial realizations can be random, are more difficult to study in a hydrodynamic framework. The first results were obtained by Dobrushin *et al.* for the one-dimensional hard rods [1], and for the one-dimensional oscillators on the lattice [6]–[8]. The main purpose of these models is to show how hydrodynamic behavior arises.

The present work continues the papers [14, 17], where as the model the harmonic crystals in $\mathbb{Z}^d$ are considered. In the harmonic approximation, the crystal is characterized by the displacements $u(z,t) \in \mathbb{R}^n$, $z \in \mathbb{Z}^d$, of the crystal atoms from their equilibrium positions. The field $u(z,t)$ is governed by a discrete wave equation. The harmonic crystals can be considered as an extension of the model of the infinite chain of one-dimensional harmonic oscillators to the many dimensional case.

The derivation of hydrodynamic equations is connected with the problem of convergence to an equilibrium measure. For harmonic crystals, such convergence was proved in [11, 12, 15]. We outline this result. The initial data are assumed to be random function with a distribution μ_0. We suppose that the measure μ_0 has zero mean value, a finite mean energy density and satisfies the mixing condition. The distribution μ_t of the random solution $u(\cdot,t)$ in the time moments $t \in \mathbb{R}$ is studied. Then the limit

$$\lim_{t\to\infty} \mu_t = \mu_\infty \tag{1.1}$$

is established, where μ_∞ is an equilibrium Gaussian measure. For one-dimensional chain of harmonic oscillators, this result has been proved by Boldrighini *et al.* [2]. The convergence to equilibrium distribution was proved also for systems described by partial differential equations [9, 10], for the crystal coupled to the scalar field [13], and for the Klein-Gordon equation coupled to a particle [16].

To derive the hydrodynamic equations we apply the special so-called *hydrodynamic limit procedure* in which the notions of hydrodynamic limit and hydrodynamic space–time rescalings play a central role. Namely, we introduce a small scale parameter $\varepsilon > 0$ giving the relation between the microscopic and macroscopic space-time scales and consider the family of the initial measures $\{\mu_0^\varepsilon, \varepsilon > 0\}$ which satisfies some conditions (see conditions **V1** and **V2** in Section 2.2 below). In particular, we assume that (i) the measures μ_0^ε are locally

homogeneous for space translations of order much less than ε^{-1} and nonhomogeneous for translations of order ε^{-1}; (ii) the covariance of μ_0^ε vanishes with distance enough quickly and uniformly in ε.

To deduce the Euler equation we use (see [14]) a *hyperbolic* (or *Euler*) *scaling*, i.e., the *microscopic* time and space variables are $t=\tau/\varepsilon$ and $z=r/\varepsilon$, where the *macroscopic* time and space variables are denoted by τ and r, respectively. Given nonzero $\tau\in\mathbb{R}$ and $r\in\mathbb{R}^d$, we study the distribution $\mu^\varepsilon_{\tau/\varepsilon,r/\varepsilon}$ of the random solution $u(z,t)$ at the space points close to $[r/\varepsilon]\in\mathbb{Z}^d$ and in time moments τ/ε. Then the limit is established,

$$\lim_{\varepsilon\to 0}\mu^\varepsilon_{\tau/\varepsilon,r/\varepsilon}=\mu^G_{\tau,r}, \tag{1.2}$$

where $\mu^G_{\tau,r}$ is a Gaussian measure (see Theorem 3.3 in [14] for harmonic crystals in the whole space $\mathbb{Z}^d$ and Theorem 2.15 in [17] for harmonic crystals in the half-space $\mathbb{Z}^d_+$). In particular, we derive the explicit formulas for covariance matrix $q_{\tau,r}(z-z')$ of the limit measure $\mu^G_{\tau,r}$. These formulas allow us to conclude that in Fourier transform the matrix function $\hat q_{\tau,r}(\theta)$, $\theta\in\mathbb{T}^d$, evolves according to the following equation:

$$\partial_\tau f(\tau,r;\theta)=i\,C(\theta)\nabla\Omega(\theta)\cdot\nabla_r f(\tau,r;\theta),\quad r\in\mathbb{R}^d,\quad \tau>0, \tag{1.3}$$

where $C(\theta)=\begin{pmatrix}0 & \Omega^{-1}(\theta)\\ -\Omega(\theta) & 0\end{pmatrix}$, and, roughly, $\Omega(\theta)$ is the dispersion relation of the harmonic crystal (for details, see Section 3.1 below). Here and below ∂_τ denotes partial differetiation with respect to a time τ, $\nabla_r f$ is the gradient of f with respect to $r\in\mathbb{R}^d$, " $\cdot$ " stands for the standard Euclidean scalar product in $\mathbb{R}^d$ (or in $\mathbb{R}^n$). The equation (1.3) should be considered as the analog of the Euler equation for our model. In [6], the similar equation was deduced in the case $d=n=1$. These results were extended to the harmonic crystals in the entire space $\mathbb{Z}^d$, $d\ge 1$, (see [14]) and in the half-space $\mathbb{Z}^d_+=\{z\in\mathbb{Z}^d: z_1>0\}$, see [17].

The main result of the given paper is the derivation of the equation for the "next approximation" to the Euler equation (1.3). To obtain the additional term of order ε in (1.3), we use a *diffusive* (or *parabolic*) scaling, that is we study the distributions of solution $u(z,t)$ in time moments of order τ/ε^2. After the appropriate change of variables we derive the equation of the Navier–Stokes type

$$\partial_\tau\hat f(\tau,r;\theta)=iC(\theta)\left(\nabla\omega(\theta)\cdot\nabla_r\hat f(\tau,r;\theta)+\frac{i\varepsilon}{2}\operatorname{tr}\left[\nabla^2\Omega(\theta)\cdot\nabla_r^2 f(\tau,r;\theta)\right]\right). \tag{1.4}$$

Here and below $\nabla_r^2 f$ stands for the matrix of second partial derivatives of f with respect to r, "tr" stands for trace. The precise statement of the result see in Theorem 3.6 and Corollary 3.7 below. In the case $d=n=1$, these results have been obtained in the work of Dobrushin *et al* [7]. Therefore, in the proof (see Section 6) we use an approach of [7] and tools of [11, 14] developed for harmonic crystals in any dimension.

In Section 3.3 we use a scaling of the form $t=\tau/\varepsilon^k$, with $k\ge 2$, $z=r/\varepsilon$, and derive the "corrections" of the higher order (i.e. of the order ε^k with $k\ge 2$) to equation (1.3).

Second part of the paper is devoted to the study of the harmonic crystals in the half-space $\mathbb{Z}^d_+=\{z\in\mathbb{Z}^d: z_1>0\}$, with zero boundary condition. For such model, we also derive the limiting "hydrodynamic" equations (of the Euler and Navier–Stokes types).

The paper is organized as follows. In Section 2 we introduce the model, impose the main conditions on harmonic potentials and initial measures μ_0^ε and give the examples of the potentials and measures μ_0^ε satisfying our conditions. In Section 3 the main results are stated. Sections 4 and 5 are devoted to the harmonic crystals in the half-space $\mathbb{Z}^d_+$. Sections 6 and 7 contain the main steps of the proof of results. The technical details of the proof are given in Appendices A–C. Appendix D is devoted to locally conserved quantities.

2 Model I: Harmonic crystals in $\mathbb{Z}^d$

We study the dynamics of the harmonic crystals in $\mathbb{Z}^d$, $d\ge 1$,

$$\left\{\begin{array}{l} \ddot v(z,t)=-\sum\limits_{z'\in\mathbb{Z}^d} V(z-z')v(z',t), \quad z\in\mathbb{Z}^d, \quad t\in\mathbb{R}, \\ v(z,0)=v_0(z), \quad \dot v(z,0)=v_1(z), \quad z\in\mathbb{Z}^d. \end{array}\right. \tag{2.1}$$

Here $v(z,t)=(v_1(z,t),\ldots,v_n(z,t))$, $v_0(z)=(v_{01}(z),\ldots,v_{0n}(z))\in\mathbb{R}^n$, and correspondingly for $v_1(z)$, $V(z)$ is the interaction (or force) matrix, $(V_{kl}(z))$, $k,l=1,\ldots,n$.

Write $X(t)=(X^0(t),X^1(t))\equiv(v(\cdot,t),\dot v(\cdot,t))$ and $X_0=(X^0_0,X^1_0)\equiv(v_0(\cdot),v_1(\cdot))$. Then (2.1) becomes

$$\dot X(t)=\mathcal{A}X(t), \quad t\in\mathbb{R}, \quad X(0)=X_0. \tag{2.2}$$

Here $\mathcal{A}=\begin{pmatrix} 0 & 1 \\ -\mathcal{V} & 0\end{pmatrix}$, where $\mathcal{V}$ is a convolution operator with the matrix kernel V, $\mathcal{V}v=\sum\limits_{z'\in\mathbb{Z}^d} V(z-z')v(z')$. Formally, (2.2) is a linear Hamiltonian system with the Hamiltonian functional

$$H(X)=\frac12\sum_{z\in\mathbb{Z}^d}|v_1(z)|^2+\frac12\sum_{z,z'\in\mathbb{Z}^d} v_0(z)\cdot V(z-z')v_0(z'), \quad X=(v_0,v_1), \tag{2.3}$$

where the kinetic energy is given by the first term, and the potential energy by the second term.

Assume that the initial data X_0 for (2.2) belong to the phase space $\mathcal{H}_\alpha$, $\alpha\in\mathbb{R}$, defined below.

Definition 2.1 *$\mathcal{H}_\alpha$ is the Hilbert space of $\mathbb{R}^n\times\mathbb{R}^n$-valued functions of $z\in\mathbb{Z}^d$ endowed with the norm $\|X\|^2_\alpha=\sum_{z\in\mathbb{Z}^d}|X(z)|^2(1+|z|^2)^\alpha<\infty$.*

2.1 Conditions on the harmonic potentials

We impose the following conditions **E1**–**E6** on the matrix V.

E1. There are positive constants C and γ such that $\|V(z)\|\le Ce^{-\gamma|z|}$ for $z\in\mathbb{Z}^d$, where $\|V(z)\|$ stands for the matrix norm.

E2. The matrix $V(z)$ is real and symmetric, i.e., $V_{lk}(-z)=V_{kl}(z)\in\mathbb{R}$, $k,l=1,\ldots,n$, $z\in\mathbb{Z}^d$.

Let $\hat{V}(\theta)$ be the Fourier transform of $V(z)$ with the convention $\hat{V}(\theta) = \sum\limits_{z\in\mathbb{Z}^d} V(z)e^{iz\cdot\theta}$, $\theta \in \mathbb{T}^d$, where $\mathbb{T}^d$ stands for the d-torus $\mathbb{R}^d/(2\pi\mathbb{Z})^d$.

Conditions **E1** and **E2** imply that $\hat{V}(\theta)$ is a real-analytic Hermitian matrix-valued function of $\theta \in \mathbb{T}^d$.

E3. The matrix $\hat{V}(\theta)$ is non-negative definite for every $\theta \in \mathbb{T}^d$.

Let us define the Hermitian non-negative definite matrix,

$$\Omega(\theta) = (\hat{V}(\theta))^{1/2} \geq 0. \tag{2.4}$$

The matrix $\Omega(\theta)$ has the eigenvalues $0 \leq \omega_1(\theta) < \omega_2(\theta) < \ldots < \omega_s(\theta)$, $s \leq n$, and the corresponding spectral projections $\Pi_\sigma(\theta)$ with multiplicity $r_\sigma = \operatorname{tr} \Pi_\sigma(\theta)$. The following lemma holds.

Lemma 2.2 *(see [11, Lemma 2.2]). Let conditions* **E1** *and* **E2** *hold. Then there exists a closed subset $\mathcal{C}_* \subset \mathbb{T}^d$ such that the following assertions hold.*
(i) The Lebesgue measure of $\mathcal{C}_*$ is zero.
(ii) The eigenvalue $\omega_\sigma(\theta)$, $\sigma = 1, \ldots, s$, has constant multiplicity in $\mathbb{T}^d \setminus \mathcal{C}_*$.
(iii) The following spectral decomposition holds: $\Omega(\theta) = \sum_{\sigma=1}^{s} \omega_\sigma(\theta)\Pi_\sigma(\theta)$, $\theta \in \mathbb{T}^d \setminus \mathcal{C}_*$, where $\Pi_\sigma(\theta)$ is a real-analytic function on $\mathbb{T}^d \setminus \mathcal{C}_*$.

For $\theta \in \mathbb{T}^d \setminus \mathcal{C}_*$, denote by $\nabla^2\omega_\sigma(\theta)$ the matrix of second partial derivatives. The next condition on V is as follows.

E4. The functions $D_\sigma(\theta) := \det(\nabla^2\omega_\sigma(\theta))$ do not vanish identically on $\mathbb{T}^d \setminus \mathcal{C}_*$, $\sigma = 1, \ldots, s$.

Let us write

$$\mathcal{C}_0 = \{\theta \in \mathbb{T}^d : \det \hat{V}(\theta) = 0\}, \quad \mathcal{C}_\sigma = \{\theta \in \mathbb{T}^d \setminus \mathcal{C}_* : D_\sigma(\theta) = 0\}, \quad \sigma = 1, \ldots, s. \tag{2.5}$$

Then the Lebesgue measure of $\mathcal{C}_\sigma$ vanishes, $\sigma = 0, 1, ..., s$ (see [11, Lemma 2.3]). Usually, the dispersion relations $\omega_k(\theta)$ satisfying the condition $\omega_k(0) = 0$ are called *acoustic*.

E5. For each $\sigma \neq \sigma'$, the identities $\omega_\sigma(\theta) \pm \omega_{\sigma'}(\theta) \equiv \text{const}_\pm$ for $\theta \in \mathbb{T}^d \setminus \mathcal{C}_*$, do not hold with $\text{const}_\pm \neq 0$.

This condition holds trivially for $n = 1$.

E6. $\|\hat{V}^{-1}(\theta)\| \in L^1(\mathbb{T}^d)$.

If $\mathcal{C}_0 = \emptyset$, then $\|\hat{V}^{-1}(\theta)\|$ is bounded, and **E6** evidently holds.

Remark 2.3 (i) Instead of condition **E1** we may assume that $|V(z)| \leq C(1 + |z|)^{-N}$ with an $N > 0$. However, in this case, we should assume in addition that there exists a set $\mathcal{K} \subseteq \mathbb{T}^d$ such that $\operatorname{mes}(\mathbb{T}^d \setminus \mathcal{K}) = 0$ and for $\theta \in \mathcal{K}$, $\omega_\sigma \in C^3(\mathcal{K})$, $\Pi_\sigma \in C(\mathcal{K})$, $\omega_\sigma(\theta) \neq 0$, $\det(\nabla^2\omega_\sigma(\theta)) \neq 0$, $\nabla\omega_\sigma(\theta) \neq 0$, $\sigma = 1, \ldots, s$. Note that if condition **E1** holds, $\mathcal{K} = \mathbb{T}^d \setminus \mathcal{C}$ with $\mathcal{C}$ defined in (6.3).

(ii) Conditions **E1**–**E6** are satisfied, in particular, in the case of the *nearest neighbor crystal* (see [11]) in which the interaction matrix $V(z)=(V_{kl}(z))_{k,l=1}^n$ is of the form

$$V_{kl}(z)=0 \ \text{ for } \ k\neq l, \quad V_{kk}(z)=\begin{cases} -\gamma_k & \text{for } \ |z|=1, \\ 2d\gamma_k+m_k^2 & \text{for } \ z=0, \\ 0 & \text{for } \ |z|\geq 2, \end{cases} \qquad k=1,\dots,n,$$

with $\gamma_k>0$ and $m_k\geq 0$. In this case, the Hamiltonian functional has a form

$$H(v_0,v_1)=\frac{1}{2}\sum_{z\in\mathbb{Z}^d}\sum_{k=1}^n\Big(|v_{1k}(z)|^2+m_k^2|v_{0k}(z)|^2+\sum_{j=1}^d\gamma_k|v_{0k}(z+e_j)-v_{0k}(z)|^2\Big),$$

$e_j=(\delta_{1j},\dots,\delta_{dj})$, and equation (2.1) becomes

$$\ddot v_k(z,t)=(\gamma_k\Delta_L-m_k^2)v_k(z,t), \quad k=1,\dots,n,$$

where Δ_L stands for the discrete Laplace operator on the lattice $\mathbb{Z}^d$,

$$\Delta_L v(z):=\sum_{j=1}^d(v(z+e_j)-2v(z)+v(z-e_j)).$$

Therefore, the eigenvalues of $\Omega(\theta)$ are

$$\tilde\omega_k(\theta)=\sqrt{2\gamma_k(1-\cos\theta_1)+...+2\gamma_k(1-\cos\theta_d)+m_k^2}\,, \quad k=1,\dots,n. \tag{2.6}$$

These eigenvalues still have to be labelled according to magnitude and degeneracy as in Lemma 2.2. Clearly, conditions **E1**–**E5** hold, and $\mathcal{C}_*=\emptyset$. If $m_k>0$ for any k, then the set $\mathcal{C}_0$ is empty and condition **E6** holds automatically. Otherwise, if $m_k=0$ for some k, then $\mathcal{C}_0=\{0\}$. In this case, **E6** is equivalent to the condition $\omega_k^{-2}(\theta)\in L^1(\mathbb{T}^d)$. Therefore, conditions **E1**–**E6** hold if either (i) $d\geq 3$ or (ii) $d=1,2$ and $m_k>0$ for any k.

Lemma 2.4 *(see [11, Proposition 2.5]) Let conditions* **E1** *and* **E2** *hold, and* $\alpha\in\mathbb{R}$. *Then*
(i) for any $X_0\in\mathcal{H}_\alpha$, *there exists a unique solution* $X(t)\in C(\mathbb{R},\mathcal{H}_\alpha)$ *to the Cauchy problem (2.2);*
(ii) for any $t\in\mathbb{R}$, *the operator* $U(t):X_0\mapsto X(t)$ *is continuous on* $\mathcal{H}_\alpha$.

The proof of Lemma 2.4 is based on the following formula for the solution $X(t)$ of problem (2.2):

$$X(t)=\sum_{z'\in\mathbb{Z}^d}\mathcal{G}_t(z-z')X_0(z'), \tag{2.7}$$

where the function $\mathcal{G}_t(z)$ has the Fourier representation

$$\mathcal{G}_t(z):=F^{-1}_{\theta\to z}[\exp(\hat{\mathcal{A}}(\theta)t)]=(2\pi)^{-d}\int_{\mathbb{T}^d}e^{-iz\cdot\theta}\exp(\hat{\mathcal{A}}(\theta)t)\,d\theta \tag{2.8}$$

with

$$\hat{\mathcal{A}}(\theta)=\begin{pmatrix} 0 & 1 \\ -\hat V(\theta) & 0\end{pmatrix}, \quad \theta\in\mathbb{T}^d. \tag{2.9}$$

2.2 The family of initial measures

Let $\varepsilon > 0$ be a small scale parameter, $\{\mu_0^\varepsilon, \varepsilon > 0\}$ be a family of initial measures. To formulate the main conditions **V1** and **V2** on the covariance of μ_0^ε, let us introduce the complex $2n \times 2n$ matrix-valued function $\mathbf{R}_0(r,z) = (\mathbf{R}_0^{ij}(r,z))_{i,j=0}^1$, $r \in \mathbb{R}^d$, $z \in \mathbb{Z}^d$, with the following properties.

I1. For every fixed $r \in \mathbb{R}^d$ and $i,j = 0,1$, the bound holds,

$$|\mathbf{R}_0^{ij}(r,z)| \le C(1+|z|)^{-\gamma}, \quad z \in \mathbb{Z}^d, \tag{2.10}$$

where C is some positive constant, $\gamma > d$.

I2. For every fixed $r \in \mathbb{R}^d$, $\hat{\mathbf{R}}_0(r,\theta)$ satisfies

$$\hat{\mathbf{R}}_0^{00}(r,\theta) \ge 0,\ \hat{\mathbf{R}}_0^{11}(r,\theta) \ge 0,\ \hat{\mathbf{R}}_0^{01}(r,\theta) = \hat{\mathbf{R}}_0^{10}(r,\theta)^*, \quad \theta \in \mathbb{T}^d.$$

I3. For every fixed $r \in \mathbb{R}^d$ and $\theta \in \mathbb{T}^d$, the matrix $\hat{\mathbf{R}}_0(r,\theta)$ is non-negative definite.

I4. For every $\theta \in \mathbb{T}^d$, $\hat{\mathbf{R}}_0^{ij}(\cdot,\theta)$ are C^d functions, and the function

$$r \to \sup_{\theta\in\mathbb{T}^d} \max_{i,j=0,1} \max_{\alpha_k=0,1,\,k=1,\dots,d} \Big| \frac{\partial^{\alpha_1+\ldots+\alpha_d}}{\partial r_1^{\alpha_1}\ldots\partial r_d^{\alpha_d}} \hat{\mathbf{R}}_0^{ij}(r,\theta)\Big|$$

is bounded uniformly on bounded sets.

To derive the equation of the Navier–Stokes type we need an additional condition **I4'**.

I4'. $\hat{\mathbf{R}}_0(\cdot,\theta) \in L^1(\mathbb{R}^d)$, $\forall\theta \in \mathbb{T}^d$, and there exist constants $C > 0$ and $N > d+3$ such that

$$\sup_{\theta\in\mathbb{T}^d} |\tilde{\mathbf{R}}_0(s,\theta)| \le C(1+|s|)^{-N}, \quad s \in \mathbb{R}^d. \tag{2.11}$$

Here by $\tilde{\mathbf{R}}_0(s,\theta)$ we denote the Fourier transform of $\hat{\mathbf{R}}_0(r,\theta)$ with respect to r:

$$\tilde{\mathbf{R}}_0(s,\theta) = F_{r\to s}[\hat{\mathbf{R}}_0(r,\theta)] = \int\limits_{\mathbb{R}^d} e^{is\cdot r}\hat{\mathbf{R}}_0(r,\theta)\,dr, \quad s \in \mathbb{R}^d, \quad \theta \in \mathbb{T}^d. \tag{2.12}$$

This condition could be weakened (see condition D' in [7] for the case $d = n = 1$). Instead of **I4'** we may assume that for each $\theta \in \mathbb{T}^d$ the function $\hat{\mathbf{R}}_0(\cdot,\theta)$ admits the representation

$$\hat{\mathbf{R}}_0(r,\theta) = (2\pi)^{-d} \int\limits_{\mathbb{R}^d} e^{-is\cdot r}\mu(\theta,ds), \tag{2.13}$$

where $\mu(\theta,ds)$ is some Borel (complex-valued) measure on $\mathbb{R}^d$, depending on the parameter $\theta \in \mathbb{T}^d$. Let $|\mu|(\theta,\cdot)$, $\theta \in \mathbb{T}^d$, be the *total variation* of the measure $\mu(\theta,\cdot)$. This means (see [26, §29]) that for any measurable set $E \subset \mathbb{R}^d$, $|\mu|(\theta,E) = \sup |\int\limits_E f(s)\mu(\theta,ds)|$, where supremum is taken over all measurable functions f such that $|f| \le 1$. We assume that there exist constants $C > 0$ and $\delta > d^2/2 + 2d + 1$ such that for any $n \in \mathbb{N}$,

$$\sup_{\theta\in\mathbb{T}^d} |\mu|(\theta,K_n) \le C(1+n)^{-\delta}, \quad \text{where } K_n = \{s \in \mathbb{R}^d : n \le |s| < n+1\}. \tag{2.14}$$

Denote by $\mathbb{E}_0^\varepsilon$ expectation with respect to the measure μ_0^ε, and by $Q_\varepsilon^{ij}(z,z')=\mathbb{E}_0^\varepsilon(X^i(z)\otimes X^j(z'))$, $z,z'\in\mathbb{Z}^d$, $i,j=0,1$, the correlation functions of μ_0^ε. Assume that $\mathbb{E}_0^\varepsilon(X(z))=0$ and functions $Q_\varepsilon^{ij}(z,z')$ satisfy the following conditions **V1** and **V2**.

V1. For any $\varepsilon>0$, $z,z'\in\mathbb{Z}^d$,

$$\left|Q_\varepsilon^{ij}(z,z')-\mathbf{R}_0^{ij}(\varepsilon z,z-z')\right|\le C\min\Big[(1+|z-z'|)^{-\gamma},\varepsilon|z-z'|\Big], \tag{2.15}$$

with the constants C, γ as in (2.10).

V2. For any $\varepsilon>0$ and all $z,z'\in\mathbb{Z}^d$, $i,j=0,1$, $|Q_\varepsilon^{ij}(z,z')|\le C(1+|z-z'|)^{-\gamma}$ with the constants C and γ as in (2.10).

Remarks (i) Condition **V1** can be formulated in the another form (see [6, 14]). Namely, for any $\varepsilon>0$ there exists an even integer N_ε such that
a) for all $M\in\mathbb{R}^d$ and $z,z'\in I_M$,

$$\left|Q_\varepsilon^{ij}(z,z')-\mathbf{R}_0^{ij}(\varepsilon M,z-z')\right|\le C\min[(1+|z-z'|)^{-\gamma},\varepsilon N_\varepsilon],$$

where C, γ are the constants from (2.10), and I_M is the cube centered at the point M with edge length N_ε, $I_M=\{z=(z_1,\dots,z_d)\in\mathbb{Z}^d:\ |z_j-M_j|\le N_\varepsilon/2,\ M=(M_1,\dots,M_d)\}$;
b) $N_\varepsilon\sim\varepsilon^{-\beta}$ as $\varepsilon\to0$, with some $\beta\in(1/2,1)$.

The formulation of **V1** in the form (2.15) is more natural and convenient for our proof.

(ii) By conditions **V1** and **V2**, $\sum\limits_{p\in\mathbb{Z}^d}e^{i\theta\cdot p}Q_\varepsilon^{ij}([r/\varepsilon+p/2],[r/\varepsilon-p/2])\to\hat{\mathbf{R}}_0^{ij}(r,\theta)$ as $\varepsilon\to0$ uniformly in $r\in\mathbb{R}^d$ and $\theta\in\mathbb{T}^d$. Here and below $[x]=([x_1],\dots,[x_d])$ for $x\in\mathbb{R}^d$ and $[x_i]$ stands for the integer part of $x_i\in\mathbb{R}^1$, $i=1,\dots,d$.

2.3 Example of initial measures μ_0^ε

We construct Gaussian initial measures μ_0^ε satisfying conditions **V1** and **V2**. At first, we introduce matrix-valued functions $q_0^{ij}(z)$, $z\in\mathbb{Z}^d$, such that $q_0^{ij}(z)=0$ for $i\ne j$, and

$$\hat q_0^{ii}(\theta)=F_{z\to\theta}[q_0^{ii}(z)]\in L^1(\mathbb{T}^d),\quad \hat q_0^{ii}(\theta)\ge0,\quad i=0,1.$$

Next, we set

$$\mathbf{R}_0^{ij}(r,z)=T(r)q_0^{ij}(z),\quad r\in\mathbb{R}^d,\quad z\in\mathbb{Z}^d, \tag{2.16}$$

where $T\in C^d(\mathbb{R}^d)$, $T(r)\ge0$, $\sup\limits_{r\in\mathbb{R}^d}\sup\limits_{|\alpha|\le d}|D^\alpha T(r)|\le C<\infty$, $|F_{r\to s}[T(r)]|\le C(1+|s|)^{-N}$ with an $N>d$. Finally, we put

$$Q_\varepsilon^{ij}(z,z')=\sqrt{T(\varepsilon z)T(\varepsilon z')}\,q_0^{ij}(z-z'),\quad z,z'\in\mathbb{Z}^d,\quad i,j=0,1. \tag{2.17}$$

By the Minlos theorem, for any $\varepsilon>0$, there exists a Borel Gaussian measure μ_0^ε on $\mathcal{H}_\alpha$, $\alpha<-d/2$, with the correlation functions $Q_\varepsilon^{ij}(z,z')$, because

$$\mathbb{E}_0^\varepsilon(\|X\|_\alpha^2)\ =\ \sum_{z\in\mathbb{Z}^d}(1+|z|^2)^\alpha\,\mathrm{tr}[Q_\varepsilon^{00}(z,z)+Q_\varepsilon^{11}(z,z)]$$

$$
\begin{aligned}
&= \sum_{z\in\mathbb{Z}^d} (1+|z|^2)^\alpha T(\varepsilon z)\,\mathrm{tr}[q_0^{00}(0)+q_0^{11}(0)] \\
&\le C(\alpha,d)\,\mathrm{tr}\int_{\mathbb{T}^d} (\hat q_0^{00}(\theta)+\hat q_0^{11}(\theta))\,d\theta \le C_1<\infty .
\end{aligned}
$$

Let us assume that there exist constants $C>0$ and $\gamma>d$ such that

$$
|q_0^{ij}(z)| \le C(1+|z|)^{-\gamma}, \quad z\in\mathbb{Z}^d. \tag{2.18}
$$

Then $Q_\varepsilon^{ij}(z,z')$ satisfy the conditions **V1** and **V2**.

Definition 2.5 *Formally, Gibbs measure g is $g(dX)=\frac{1}{Z}e^{-\frac{\beta}{2}\sum_z H(X)}\prod_{z\in\mathbb{Z}^d} dX(z)$, where $H(X)$ is defined in (2.3), $\beta=T^{-1}$, $T\ge 0$ is the corresponding absolute temperature. We define the Gibbs measure g on $\mathcal{H}_\alpha$, $\alpha<-d/2$, as the Gaussian measure with the correlation matrices defined by their Fourier transform as $\hat q^{00}(\theta)=T\hat V^{-1}(\theta)$, $\hat q^{11}(\theta)=TI$, $\hat q^{01}(\theta)=\hat q^{10}(\theta)=0$, where I stands for the unit matrix in $\mathbb{R}^n$.*

Let $\hat q_0^{00}(\theta)=\hat V^{-1}(\theta)$, $\hat q_0^{11}(\theta)=I$, $\theta\in\mathbb{T}^d$. Then the functions $\mathbf{R}_0^{ij}(r,z)$ defined in (2.16) are correlation matrices of the Gibbs measures g_r, $r\in\mathbb{R}^d$, with $\beta=1/T(r)$. If we assume, in addition, that $\mathcal{C}_0=\emptyset$, i.e. $\det\hat V(\theta)\ne 0$, $\forall\theta\in\mathbb{T}^d$, then the bound (2.18) holds, and $Q_\varepsilon^{ij}(z,z')$ defined in (2.17) satisfy the conditions **V1** and **V2**.

3 Main results

Definition 3.1 *(i) μ_t^ε is a Borel probability measure on $\mathcal{H}_\alpha$ which gives the distribution of the random solution $X(t)$, $\mu_t^\varepsilon(B)=\mu_0^\varepsilon(U(-t)B)$, where $B\in\mathcal{B}(\mathcal{H}_\alpha)$ and $t\in\mathbb{R}$.*

(ii) The correlation functions of the measure μ_t^ε are defined by

$$
Q_{\varepsilon,t}^{ij}(z,z') = \mathbb{E}_t^\varepsilon\left(X^i(z)\otimes X^j(z')\right) = \mathbb{E}_0^\varepsilon(X^i(z,t)\otimes X^j(z',t)), \quad i,j=0,1, \quad z,z'\in\mathbb{Z}^d,
$$

where $\mathbb{E}_t^\varepsilon$ stands for expectation with respect to the measure μ_t^ε, and $X^i(z,t)$ are the components of the random solution $X(t)=(X^0(\cdot,t),X^1(\cdot,t))$ to problem (2.2).

(iii) Let T_h, $h\in\mathbb{Z}^d$, be the group of space translations: $T_hX(z)=X(z-h)$, $z\in\mathbb{Z}^d$. For $\tau\ne 0$, $r\in\mathbb{R}^d$, and $\kappa>0$, the measures $\mu^\varepsilon_{\tau/\varepsilon^\kappa,r/\varepsilon}$ are defined by the rule

$$
\mu^\varepsilon_{\tau/\varepsilon^\kappa,r/\varepsilon}(B)=\mu^\varepsilon_{\tau/\varepsilon^\kappa}(T_{[r/\varepsilon]}B), \quad \textit{where } B\in\mathcal{B}(\mathcal{H}_\alpha).
$$

Remarks (i) In addition to conditions **V1** and **V2**, let us assume that the measures μ_0^ε satisfy the *mixing condition*. To formulate this condition, denote by $\sigma(\mathcal{A})$, $\mathcal{A}\subset\mathbb{Z}^d$, the σ-algebra on $\mathcal{H}_\alpha$ generated by $X_0(z)$ with $z\in\mathcal{A}$. Define the Ibragimov mixing coefficients of the probability measure μ_0^ε on $\mathcal{H}_\alpha$ by the rule

$$
\varphi_\varepsilon(r) = \sup_{\substack{\mathcal{A},\mathcal{B}\subset\mathbb{Z}^d \\ \mathrm{dist}(\mathcal{A},\,\mathcal{B})\ge r}} \ \sup_{\substack{A\in\sigma(\mathcal{A}),\,B\in\sigma(\mathcal{B}) \\ \mu_0^\varepsilon(B)>0}} \frac{|\mu_0^\varepsilon(A\cap B)-\mu_0^\varepsilon(A)\mu_0^\varepsilon(B)|}{\mu_0^\varepsilon(B)}.
$$

A measure μ_0^ε is said to satisfy the *strong uniform Ibragimov mixing condition* if $\varphi_\varepsilon(r)\to 0$ as $r\to\infty$. Moreover, we assume that $\forall r\in\mathbb{R}^d$, $\sup_{\varepsilon>0}\varphi_\varepsilon(r)\le C(1+r)^{-2\gamma}$, with the constant γ as in (2.10). Note that the last bound on $\varphi_\varepsilon(r)$ implies condition **V2**.

Then for $\tau\ne 0$, $r\in\mathbb{R}^d$, the measures $\mu^\varepsilon_{\tau/\varepsilon,r/\varepsilon}$ converge weakly to a limit measure $\mu^G_{\tau,r}$ on the space $\mathcal{H}_\alpha$, $\alpha<-d/2$. By definition, this means that

$$\lim_{\varepsilon\to 0}\int f(X)\mu^\varepsilon_{\tau/\varepsilon,r/\varepsilon}(dX)=\int f(X)\mu^G_{\tau,r}(dX)$$

for any bounded continuous functional f on $\mathcal{H}_\alpha$. Moreover, the limit measure $\mu^G_{\tau,r}$ is a Gaussian measure on $\mathcal{H}_\alpha$ (see Theorem 3.3 in [14]).

(ii) Let $\kappa<1$. Then for $\tau\ne 0$, $r\in\mathbb{R}^d$, the following limit holds, $\mu^\varepsilon_{\tau/\varepsilon^\kappa,r/\varepsilon}\rightharpoondown\mu_r$ as $\varepsilon\to 0$ in the sense of weak convergence on $\mathcal{H}_\alpha$, $\alpha<-d/2$. Moreover, the limit measure μ_r is Gaussian, its correlation matrix does not depend on τ and has a form (in Forier transform)

$$\frac12\sum_{\sigma=1}^s\Pi_\sigma(\theta)\Big(\hat{\mathbf{R}}_0(r,\theta)+C_\sigma(\theta)\hat{\mathbf{R}}_0(r,\theta)C^*_\sigma(\theta)\Big)\Pi_\sigma(\theta),\quad \theta\in\mathbb{T}^d\setminus\mathcal{C}_*,$$

where $C_\sigma(\theta)$ is defined in (3.4).

3.1 Equation of Euler type

In this subsection we put $\kappa=1$. Let us introduce the matrix $q_{\tau,r}(z)$, $z\in\mathbb{Z}^d$, $\tau\in\mathbb{R}$, $r\in\mathbb{R}^d$. In Fourier space,

$$\begin{aligned}\hat q_{\tau,r}(\theta) &= \frac14\sum_{\sigma=1}^s\Pi_\sigma(\theta)\Big[\sum_\pm(I\pm iC_\sigma(\theta))\hat{\mathbf{R}}_0(r\pm\nabla\omega_\sigma(\theta)\tau,\theta)(I\mp iC^*_\sigma(\theta))\Big]\Pi_\sigma(\theta) &(3.1)\\ &= \sum_{\sigma=1}^s\Pi_\sigma(\theta)(\mathbf{M}^\sigma_+(\tau,r;\theta)+i\,\mathbf{M}^\sigma_-(\tau,r;\theta))\Pi_\sigma(\theta),\quad\theta\in\mathbb{T}^d\setminus\mathcal{C}_*\,, &(3.2)\end{aligned}$$

where $\Pi_\sigma(\theta)$ is the spectral projection introduced in Lemma 2.2 (iii),

$$\begin{aligned}\mathbf{M}^\sigma_+(\tau,r;\theta) &= \tfrac12(R^\sigma_+(\tau,r;\theta)+C_\sigma(\theta)R^\sigma_+(\tau,r;\theta)C^*_\sigma(\theta)),\\ \mathbf{M}^\sigma_-(\tau,r;\theta) &= \tfrac12(C_\sigma(\theta)R^\sigma_-(\tau,r;\theta)-R^\sigma_-(\tau,r;\theta)C^*_\sigma(\theta)),\end{aligned}\qquad(3.3)$$

$$C_\sigma(\theta)=\left(\begin{array}{cc}0 & \omega_\sigma^{-1}(\theta)\\ -\omega_\sigma(\theta) & 0\end{array}\right),\quad \sigma=1,\dots,s,\qquad(3.4)$$

$$R^\sigma_\pm(\tau,r;\theta)=\frac12\Big(\hat{\mathbf{R}}_0(r+\nabla\omega_\sigma(\theta)\tau,\theta)\pm\hat{\mathbf{R}}_0(r-\nabla\omega_\sigma(\theta)\tau,\theta)\Big).\qquad(3.5)$$

Theorem 3.2 *(see [14, Theorem 4.1]) Let the conditions* **V1**, **V2** *and* **E1**–**E6** *hold. Then for any* $r\in\mathbb{R}^d$, $z,z'\in\mathbb{Z}^d$, $\tau\ne 0$, *the correlation functions of measures* $\mu^\varepsilon_{\tau/\varepsilon,r/\varepsilon}$ *converge to a limit,*

$$\lim_{\varepsilon\to 0}Q_{\varepsilon,\tau/\varepsilon}([r/\varepsilon]+z,[r/\varepsilon]+z')=q_{\tau,r}(z-z').$$

Corollary 3.3 *Write* $f_\sigma(\tau,r;\theta)=\Pi_\sigma(\theta)\hat q_{\tau,r}(\theta)\Pi_\sigma(\theta)$, $\sigma=1,\dots,s$. *Then the function* $f_\sigma(\tau,r;\theta)$ *satisfies the "hydrodynamic" Euler type equation:*

$$\partial_\tau f_\sigma(\tau,r;\theta) = iC_\sigma(\theta)\nabla\omega_\sigma(\theta)\cdot\nabla_r f_\sigma(\tau,r;\theta),\quad r\in\mathbb{R}^d,\quad \tau>0, \tag{3.6}$$

$$f_\sigma(\tau,r;\theta)|_{\tau=0} = \frac{1}{2}\Pi_\sigma(\theta)\Big(\hat{\mathbf{R}}_0(r,\theta)+C_\sigma(\theta)\hat{\mathbf{R}}_0(r,\theta)C^*_\sigma(\theta)\Big)\Pi_\sigma(\theta). \tag{3.7}$$

Remarks 3.4 (i) Note that $\hat q_{\tau,r}(\theta)=(\hat q^{ij}_{\tau,r}(\theta))^1_{i,j=0}$ satisfies the *equilibrium condition*, i.e., $\hat q^{11}_{\tau,r}(\theta)=\Omega^2(\theta)\hat q^{00}_{\tau,r}(\theta)$, $\hat q^{01}_{\tau,r}(\theta)=-\hat q^{10}_{\tau,r}(\theta)$. Moreover, $\hat q^{ii}_{\tau,r}(\theta)^*=\hat q^{ii}_{\tau,r}(\theta)\ge 0$, $\hat q^{01}_{\tau,r}(\theta)^*=\hat q^{10}_{\tau,r}(\theta)$.

(ii) In the case when $\kappa\in[1,2)$, the correlation matrices of measures $\mu^\varepsilon_{\tau/\varepsilon^\kappa,r/\varepsilon}$ converge as $\varepsilon\to 0$ to the same matrices $q_{\tau,r}(z-z')$ as in Theorem 3.2. This result can be proved by the similar way as Theorem 3.2.

Theorem 3.2 and Corollary 3.3 can be rewritten in the terms of the Wigner matrices. Write $(\mathcal{V}^k v)(x)=F^{-1}_{\theta\to x}[\hat V^k(\theta)\hat v(\theta)]$, $k\in\mathbb{R}$, and introduce the complex-valued field

$$a(x)=\frac{1}{\sqrt2}\Big(\mathcal{V}^{1/4}v_0(x)+i\mathcal{V}^{-1/4}v_1(x)\Big)\in\mathbb{C}^n,\quad x\in\mathbb{Z}^d, \tag{3.8}$$

with complex conjugate field $a(x)^*=(1/\sqrt2)\left(\mathcal{V}^{1/4}v_0(x)-i\mathcal{V}^{-1/4}v_1(x)\right)$. Define the scaled $n\times n$ Wigner matrix as

$$W^\varepsilon(\tau,r;\theta)=\sum_{y\in\mathbb{Z}^d}e^{i\theta\cdot y}\,\mathbb{E}^\varepsilon_{\tau/\varepsilon}(a^*([r/\varepsilon+y/2])\otimes a([r/\varepsilon-y/2])). \tag{3.9}$$

By properties of μ^ε_0, the following limit exists

$$\begin{aligned}\lim_{\varepsilon\to0}W^\varepsilon(0,r;\theta) &= \frac12\Big(\Omega^{1/2}\hat{\mathbf{R}}^{00}_0(r,\theta)\Omega^{1/2}+\Omega^{-1/2}\hat{\mathbf{R}}^{11}_0(r,\theta)\Omega^{-1/2}\\ &\quad +i\Omega^{1/2}\hat{\mathbf{R}}^{01}_0(r,\theta)\Omega^{-1/2}-i\Omega^{-1/2}\hat{\mathbf{R}}^{10}_0(r,\theta)\Omega^{1/2}\Big)\equiv W(0,r;\theta).\end{aligned} \tag{3.10}$$

Theorem 3.5 *(see [14, Theorem 3.2]) Let the conditions* **V1**, **V2** *and* **E1**–**E6** *hold. Then for any* $r\in\mathbb{R}^d$ *and* $\tau\ne0$ *the following limit exists*

$$\lim_{\varepsilon\to0}W^\varepsilon(\tau,r;\theta)=W^{\rm p}(\tau,r;\theta)\quad \textit{(in the sense of distributions)},$$

where

$$W^{\rm p}(\tau,r;\theta)=\sum_{\sigma=1}^s\Pi_\sigma(\theta)W(0,r-\tau\nabla\omega_\sigma(\theta);\theta)\Pi_\sigma(\theta). \tag{3.11}$$

Remarks (i) The limit correlation matrices $\hat q_{\tau,r}(\theta)=(\hat q^{ij}_{\tau,r}(\theta))^1_{i,j=0}$ (see (3.2)) are expressed by $W^p(\tau,r;\theta)$ as

$$\begin{aligned}\Omega(\theta)\hat q^{00}_{\tau,r}(\theta) &= \Omega(\theta)^{-1}\hat q^{11}_{\tau,r}(\theta)=\tfrac12(W^p(\tau,r;\theta)+W^p(\tau,r;-\theta)^*),\\ \hat q^{01}_{\tau,r}(\theta) &= -\hat q^{10}_{\tau,r}(\theta)=-\tfrac{i}{2}(W^p(\tau,r;\theta)-W^p(\tau,r;-\theta)^*).\end{aligned}$$

(ii) The matrix $f_\sigma(\tau,r;\theta)\equiv\Pi_\sigma(\theta)W^p(\tau,r;\theta)\Pi_\sigma(\theta)$, $\sigma=1,\dots,s$, satisfies the *energy transport* equation (see [14, p.656])

$$\partial_\tau f_\sigma(\tau,r;\theta)+\nabla\omega_\sigma(\theta)\cdot\nabla_r f_\sigma(\tau,r;\theta)=0,\quad \tau>0,\quad r\in\mathbb{R}^d,$$

with the initial condition $f_\sigma(\tau,r;\theta)|_{\tau=0}=\Pi_\sigma(\theta)W(0,r;\theta)\Pi_\sigma(\theta)$, $r\in\mathbb{R}^d$.

3.2 Navier–Stokes equation

In this subsection we study the behaviour (as $\varepsilon\to 0$) of the correlation functions of $\mu^\varepsilon_{\tau/\varepsilon^\kappa,r/\varepsilon}$ for $\kappa=2$, and obtain the next term of the decomposition in ε to equation (3.6). Let us introduce the matrix $q^\varepsilon_{\tau,r}(z)$, $z\in\mathbb{Z}^d$, $r\in\mathbb{R}^d$, $\tau\ne 0$, $\varepsilon>0$, which has the following form (in the Fourier transform)

$$\hat q^\varepsilon_{\tau,r}(\theta)=\frac14\sum_{\sigma=1}^s\Pi_\sigma(\theta)\Big[\sum_{\pm}(I\pm iC_\sigma(\theta))A^\pm_{\varepsilon,\sigma}(\tau,r;\theta)(I\mp iC^*_\sigma(\theta))\Big]\Pi_\sigma(\theta),\quad \theta\in\mathbb{T}^d\setminus\mathcal{C}_*,\tag{3.12}$$

where matrices $C_\sigma(\theta)$ are defined in (3.4),

$$A^\pm_{\varepsilon,\sigma}(\tau,r;\theta)=\int\limits_{\mathbb{R}^d}\hat{\mathbf{R}}_0(r\pm\nabla\omega_\sigma(\theta)\tau/\varepsilon-x,\theta)K^\pm_\sigma(\tau,x,\theta)\,dx,\tag{3.13}$$

$$K^\pm_\sigma(\tau,x,\theta):=F^{-1}_{y\to x}[e^{\mp i(\tau/2)y\cdot(\nabla^2\omega_\sigma(\theta))y}],\quad x\in\mathbb{R}^d\tag{3.14}$$

(see also formula (6.25)).

Theorem 3.6 *Let conditions* **I1**–**I4'**, **V1**, **V2** *and* **E1**–**E6** *hold. Then for any* $\tau\ne0$, $r\in\mathbb{R}^d$, $z,z'\in\mathbb{Z}^d$, *the correlation functions of measures* $\mu^\varepsilon_{\tau/\varepsilon^2,r/\varepsilon}$ *have the following asymptotics*

$$\lim_{\varepsilon\to0}\big(Q_{\varepsilon,\tau/\varepsilon^2}([r/\varepsilon]+z,[r/\varepsilon]+z')-q^\varepsilon_{\tau,r}(z-z')\big)=0,\tag{3.15}$$

where the matrix $q^\varepsilon_{\tau,r}(z)=F^{-1}_{\theta\to z}[\hat q^\varepsilon_{\tau,r}(\theta)]$ *is defined by (3.12).*

We omit the proof of this theorem since it can be proved by the similar technique as Theorem 5.6, below.

Remark. Note that $\hat q^\varepsilon_{\tau,r}(\theta)$ satisfies the equilibrium condition (see Remarks 3.4 (i)).

Set $\tau=\varepsilon t$. It follows from formulas (3.12)–(3.14) that

$$A^\pm_{\varepsilon,\sigma}(\varepsilon t,r;\theta)|_{\varepsilon=0}=\hat{\mathbf{R}}_0(r\pm\nabla\omega_\sigma(\theta)t,\theta).$$

Hence, $\hat q^\varepsilon_{\varepsilon t,r}(\theta)|_{\varepsilon=0}=\hat q_{t,r}(\theta)$, where $\hat q_{t,r}(\theta)$ is defined in (3.1). Denote

$$\nabla^k\omega_\sigma(\theta)\cdot\nabla^k_r f(r):=\sum_{i_1,\dots,i_k=1}^d\frac{\partial^k\omega_\sigma(\theta)}{\partial\theta_{i_1}\dots\partial\theta_{i_k}}\,\frac{\partial^k f}{\partial r_{i_1}\dots\partial r_{i_k}},\quad k\in\mathbb{N}.\tag{3.16}$$

Corollary 3.7 *Let* $r\in\mathbb{R}^d$, $t\in\mathbb{R}$, $\theta\in\mathbb{T}^d\setminus\mathcal{C}_*$. *It follows from formulas (3.12)–(3.14) that for each* $\sigma=1,\dots,s$, *the matrix-valued function* $f^\varepsilon_\sigma(t,r;\theta)\equiv\Pi_\sigma(\theta)\hat q^\varepsilon_{\varepsilon t,r}(\theta)\Pi_\sigma(\theta)$ *evolves according to the following (Navier–Stokes type) equation*

$$\partial_t f^\varepsilon_\sigma(t,r;\theta)=iC_\sigma(\theta)\Big(\nabla\omega_\sigma(\theta)\cdot\nabla_r+\frac{i\varepsilon}{2}\nabla^2\omega_\sigma(\theta)\cdot\nabla^2_r\Big)f^\varepsilon_\sigma(t,r;\theta),\quad t\in\mathbb{R},\quad r\in\mathbb{R}^d,\tag{3.17}$$

with the initial condition (3.7).

3.3 Corrections of the higher order

To obtain the "corrections" of order ε^2, ε^3,... to equation (3.17), it is neccesary to study the behaviour (as $\varepsilon\to 0$) of the correlation functions of measures $\mu^\varepsilon_{\tau/\varepsilon^\kappa,r/\varepsilon}$ with $\kappa>2$.

Theorem 3.8 *Let $\kappa\ge 2$, $r\in\mathbb{R}^d$, $\tau>0$. Then $\forall z,z'\in\mathbb{Z}^d$,*

$$Q_{\varepsilon,\tau/\varepsilon^\kappa}([r/\varepsilon]+z,[r/\varepsilon]+z')-q^{\varepsilon,[\kappa]}_{\tau,r}(z-z')\to 0,\quad \varepsilon\to 0.$$

The matrix $q^{\varepsilon,k}_{\tau,r}(z)$ $(k=2,3,\dots)$ has the following form (in the Fourier transform)

$$\hat q^{\varepsilon,k}_{\tau,r}(\theta)=\frac14\sum_{\sigma=1}^s\Pi_\sigma(\theta)\Big[\sum_{\pm}(I\pm iC_\sigma(\theta))A^{\pm,k}_{\varepsilon,\sigma}(\tau,r;\theta)(I\mp iC^*_\sigma(\theta))\Big]\Pi_\sigma(\theta),$$

where

$$A^{\pm,k}_{\varepsilon,\sigma}(\tau,r;\theta)=\int\limits_{\mathbb{R}^d}\hat{\mathbf{R}}_0(r\pm\nabla\omega_\sigma(\theta)\tau/\varepsilon^{k-1}-x,\theta)K^{\pm,k}_{\varepsilon,\sigma}(\tau,x,\theta)\,dx,\quad \theta\in\mathbb{T}^d\setminus\mathcal{C},$$

$$K^{\pm,k}_{\varepsilon,\sigma}(\tau,x,\theta)=F^{-1}_{y\to x}\Big[\exp\Big\{\mp i\frac{\tau}{\varepsilon^{k-2}}(\frac{\nabla^2\omega_\sigma(\theta)\cdot y^2}{2!}+\ldots+\frac{\nabla^k\omega_\sigma(\theta)\cdot y^k}{k!})\Big\}\Big],\ \ x,y\in\mathbb{R}^d.$$

Here, by definition,

$$\nabla^k\omega_\sigma(\theta)\cdot y^k:=\sum_{i_1,\dots,i_k=1}^d\frac{\partial^k\omega_\sigma(\theta)}{\partial\theta_{i_1}\dots\partial\theta_{i_k}}\,y_{i_1}\dots y_{i_k},\quad y=(y_1,\dots,y_d)\in\mathbb{R}^d.$$

Note that the matrix $\hat q^{\varepsilon,2}_{\tau,r}(\theta)$ coincides with $\hat q^{\varepsilon}_{\tau,r}(\theta)$ from formula (3.12). The proof of Theorem 3.8 are similar to the proof of Theorem 3.6.

Set $\tau=\varepsilon^{k-1}t$, $k\ge 2$. Then $\partial_t A^{\pm,k}_{\varepsilon,\sigma}(\varepsilon^{k-1}t,r;\theta)=\pm P^k_\varepsilon(\theta,\partial_r)A^{\pm,k}_{\varepsilon,\sigma}(\varepsilon^{k-1}t,r;\theta)$, where

$$P^k_\varepsilon(\theta,\partial_r):=\sum_{p=1}^k\frac{(i\varepsilon)^{p-1}}{p!}\nabla^p\omega_\sigma(\theta)\cdot\nabla^p_r$$

(see notation (3.16)). Therefore, the matrix $\Pi_\sigma(\theta)\hat q^{\varepsilon,k}_{\varepsilon^{k-1}t,r}(\theta)\Pi_\sigma(\theta)$ (denote its by $f^\varepsilon_\sigma(t,r;\theta)$) is the solution of the following equation

$$\partial_t f^\varepsilon_\sigma(t,r;\theta)=iC_\sigma(\theta)P^k_\varepsilon(\theta,\partial_r)f^\varepsilon_\sigma(t,r;\theta),\quad t>0,\quad r\in\mathbb{R}^d,\tag{3.18}$$

with the initial condition (3.7).

Denote by f^{ij}_σ, $i,j=0,1$, the elements of the matrix $f^\varepsilon_\sigma(t,r;\theta)$. Then $f^{11}_\sigma=\omega^2_\sigma(\theta)f^{00}_\sigma$, $f^{10}_\sigma=-f^{01}_\sigma$, and the equations (3.18) can be rewritten in the form

$$\partial_t f^{00}_\sigma=(-i/\omega_\sigma(\theta))P^k_\varepsilon(\theta,\partial_r)f^{01}_\sigma,\quad \partial_t f^{01}_\sigma=i\omega_\sigma(\theta)P^k_\varepsilon(\theta,\partial_r)f^{00}_\sigma.$$

4 Model II: Harmonic crystals in the half-space $\mathbb{Z}^d_+$

We study the dynamics of the harmonic crystals in $\mathbb{Z}^d_+$, $d\ge 1$,

$$\ddot u(z,t)=-\sum_{z'\in\mathbb{Z}^d_+}\left(V(z-z')-V(z-\tilde z')\right)u(z',t),\quad z\in\mathbb{Z}^d_+,\quad t\in\mathbb{R},\tag{4.1}$$

with zero boundary condition,

$$u(z,t)|_{z_1=0}=0,\tag{4.2}$$

and with the initial data

$$u(z,0)=u_0(z),\quad \dot u(z,0)=u_1(z),\quad z\in\mathbb{Z}^d_+.\tag{4.3}$$

Here $\mathbb{Z}^d_+=\{z\in\mathbb{Z}^d:\ z_1>0\}$, $\tilde z=(-z_1,z_2,\dots,z_d)$. For convenience, we assume that $u_0(z)=u_1(z)=0$ for $z_1=0$.

Write $Y(t)=(Y^0(t),Y^1(t))\equiv(u(\cdot,t),\dot u(\cdot,t))$ and $Y_0=(Y^0_0,Y^1_0)\equiv(u_0(\cdot),u_1(\cdot))$. Then (4.1)–(4.3) becomes the evolution equation

$$\dot Y(t)=\mathcal{A}_+Y(t),\quad t\in\mathbb{R},\ z\in\mathbb{Z}^d_+,\quad Y^0(t)|_{z_1=0}=0,\quad Y(0)=Y_0.\tag{4.4}$$

Here $\mathcal{A}_+=\left(\begin{array}{cc}0&1\\-\mathcal{V}_+&0\end{array}\right)$ with $\mathcal{V}_+u(z):=\sum\limits_{z'\in\mathbb{Z}^d_+}(V(z-z')-V(z-\tilde z'))u(z')$.

Let us assume that

$$V(z)=V(\tilde z),\quad \text{where } \tilde z=(-z_1,\bar z),\quad \bar z=(z_2,\dots,z_d)\in\mathbb{Z}^{d-1}.\tag{4.5}$$

Then the solution to problem (4.4) can be represented as the restriction of the solution to the Cauchy problem (2.1) with odd initial data on the half-space. More exactly, assume that the initial data $X_0(z)$ form an odd function with respect to $z_1\in\mathbb{Z}^1$, i.e., let $X_0(z)=-X_0(\tilde z)$. Then the solution $v(z,t)$ of (2.1) is also an odd function with respect to $z_1\in\mathbb{Z}^1$. Restrict the solution $v(z,t)$ to the domain $\mathbb{Z}^d_+$ and set $u(z,t)=v(z,t)|_{z_1\ge 0}$. Then $u(z,t)$ is the solution to problem (4.1) with the initial data $Y_0(z)=X_0(z)|_{z_1\ge0}$.

Assume that the initial data Y_0 for (4.4) belong to the phase space $\mathcal{H}_{\alpha,+}$, $\alpha\in\mathbb{R}$, defined below.

Definition 4.1 *$\mathcal{H}_{\alpha,+}$ is the Hilbert space of $\mathbb{R}^n\times\mathbb{R}^n$-valued functions of $z\in\mathbb{Z}^d_+$ endowed with the norm* $\|Y\|^2_{\alpha,+}=\sum_{z\in\mathbb{Z}^d_+}|Y(z)|^2(1+|z|^2)^\alpha<\infty.$

In addition, it is assumed that the initial data vanish ($Y_0=0$) at $z_1=0$.

Lemma 4.2 *(see [15, Corollary 2.4]) Let conditions* **E1** *and* **E2** *hold. Choose some $\alpha\in\mathbb{R}$. Then (i) for any $Y_0\in\mathcal{H}_{\alpha,+}$, there exists a unique solution $Y(t)\in C(\mathbb{R},\mathcal{H}_{\alpha,+})$ to the mixed problem (4.4);*
(ii) the operator $U_+(t):Y_0\mapsto Y(t)$ is continuous on $\mathcal{H}_{\alpha,+}$.

Lemma 4.2 follows from Lemma 2.4 since the solution $Y(t)$ of (4.4) admits the representation

$$Y(t)=\sum_{z'\in\mathbb{Z}^d_+}\mathcal{G}_{t,+}(z,z')Y_0(z'),\quad z\in\mathbb{Z}^d_+,\tag{4.6}$$

$$\text{where}\quad \mathcal{G}_{t,+}(z,z'):=\mathcal{G}_t(z-z')-\mathcal{G}_t(z-\tilde z'),\tag{4.7}$$

and $\mathcal{G}_t(z)$ is defined in (2.8).

4.1 The family of the initial measures

Let us introduce the complex $2n\times 2n$ matrix-valued function $R(r,x,y)=(R^{ij}(r,x,y))_{i,j=0}^{1}$, $r\in\mathbb{R}^d$, $x,y\in\mathbb{Z}^d_+$, with the following properties **(a)**–**(d)**.

(a) $R(r,x,y)=0$ for $x_1=0$ or $y_1=0$. The $n\times n$ matrix-valued functions $R^{ij}(r,x,y)$ have the form

$$R^{ij}(r,x,y)=\mathbf{R}^{ij}(r,x_1,y_1,\bar{x}-\bar{y}),\quad \text{where } x=(x_1,\bar{x}),\quad y=(y_1,\bar{y}),\quad i,j=0,1.$$

Moreover, uniformly in $r\in\mathbb{R}^d$,

$$\lim_{y_1\to+\infty}\mathbf{R}^{ij}(r,y_1+z_1,y_1,\bar{z})=\mathbf{R}_0^{ij}(r,z),\quad z=(z_1,\bar{z})\in\mathbb{Z}^d,\quad i,j=0,1, \tag{4.8}$$

where the matrix $\mathbf{R}_0(r,z)=(\mathbf{R}_0^{ij}(r,z))_{i,j=0}^{1}$ satisfies conditions **I1**–**I4** (see Section 2.2).

(b) For every fixed $r\in\mathbb{R}^d$ and $i,j=0,1$, the bound holds,

$$|R^{ij}(r,x,y)|\le C(1+|x-y|)^{-\gamma},\quad x,y\in\mathbb{Z}^d_+, \tag{4.9}$$

with the same constants C and γ as in (2.10).

(c) For every fixed $r\in\mathbb{R}^d$, the matrix-valued function $R(r,x,y)$ satisfies

$$R^{ii}(r,\cdot,\cdot)\ge 0,\quad R^{ij}(r,x,y)=(R^{ji}(r,y,x))^T,\quad x,y\in\mathbb{Z}^d_+.$$

(d) For every $x,y\in\mathbb{Z}^d_+$, $R^{ij}(\cdot,x,y)$, $i,j=0,1$, are C^1 functions.

To derive the equation of the Navier-Stokes type we need the additional condition **(d')** in the case when $d>1$.

(d') Let $d>1$. For every fixed $x,y\in\mathbb{Z}^d_+$,

$$\sup_{\bar{r}=(r_2,\dots,r_d)\in\mathbb{R}^{d-1}}|R(r_1,\bar{r},x,y)-R(0,\bar{r},x,y)|\to 0\quad\text{as } r_1\to 0. \tag{4.10}$$

Moreover, we assume that for every fixed $x,y\in\mathbb{Z}^d_+$,

$$|\tilde{R}(0,\bar{s},x,y)|\le C(1+|\bar{s}|)^{-N},\quad \bar{s}\in\mathbb{R}^{d-1}, \tag{4.11}$$

with $N>d+2$. Here by $\tilde{R}(0,\bar{s},x,y)$ we denote the Fourier transform of $R(0,\bar{r},x,y)$ w.r.t. $\bar{r}\in\mathbb{R}^{d-1}$,

$$\tilde{R}(0,\bar{s},x,y)=\int\limits_{\mathbb{R}^{d-1}}e^{i\bar{s}\bar{r}}R(0,\bar{r},x,y)\,d\bar{r},\quad \bar{s}\in\mathbb{R}^{d-1}.$$

Let $\{\mu_0^\varepsilon,\varepsilon>0\}$ be a family of initial measures on $\mathcal{H}_{\alpha,+}$ and $\mathbb{E}_0^\varepsilon$ stand for expectation with respect to the measure μ_0^ε. Assume that $\mathbb{E}_0^\varepsilon(Y(x))=0$ and define the covariance $Q_\varepsilon^{ij}(x,x')=\mathbb{E}_0^\varepsilon(Y^i(x)\otimes Y^j(x'))$, $x,x'\in\mathbb{Z}^d_+$, $i,j=0,1$.

The family of measures $\{\mu_0^\varepsilon,\varepsilon>0\}$ satisfies the following conditions **V1'** and **V2'**.

V1'. For any $\varepsilon>0$, $x,x'\in\mathbb{Z}^d_+$,

$$\left|Q_\varepsilon^{ij}(x,x')-R^{ij}(\varepsilon x,x,x')\right|\le C\min[(1+|x-x'|)^{-\gamma},\varepsilon|x-y|], \tag{4.12}$$

with the constants C and γ as in (4.9).

V2'. For any $\varepsilon>0$ and all $x,x'\in\mathbb{Z}^d_+$, $i,j=0,1$, $|Q_\varepsilon^{ij}(x,x')|\le C(1+|x-x'|)^{-\gamma}$ with the constants C, γ as in (4.9).

5 Main results in the half-space

Definition 5.1 *(i)* μ_t^ε *is a Borel probability measure on* $\mathcal{H}_{\alpha,+}$ *which gives the distribution of* $Y(t)$, $\mu_t^\varepsilon(B)=\mu_0^\varepsilon(U_+(-t)B)$, *where* $B\in\mathcal{B}(\mathcal{H}_{\alpha,+})$ *and* $t\in\mathbb{R}$.
(ii) The correlation functions of the measure μ_t^ε *are defined by*

$$Q_{\varepsilon,t}^{ij}(x,y)=\int \big(Y^i(x)\otimes Y^j(y)\big)\,\mu_t^\varepsilon(dY)=\mathbb{E}_0^\varepsilon(Y^i(x,t)\otimes Y^j(y,t)),\quad i,j=0,1,\quad x,y\in\mathbb{Z}_+^d.$$

Here $Y^i(x,t)$ *are the components of the random solution* $Y(t)=(Y^0(\cdot,t),Y^1(\cdot,t))$ *to the problem (4.4).*

5.1 Euler limit

At first, we introduce the matrix $g_{\tau,r}(z)$, $z\in\mathbb{Z}^d$, $r\in\mathbb{R}^d$, $\tau\neq 0$, by the Fourier transform,

$$\hat{g}_{\tau,r}(\theta)=\frac{1}{4}\sum_{\sigma=1}^{s}\Pi_\sigma(\theta)\Big[\sum_{\pm}(I\pm iC_\sigma(\theta))\hat{\mathbf{R}}_0(r\pm\nabla\omega_\sigma(\theta)\tau,\theta)\chi^\pm_{\tau,r_1}(\theta)(I\mp iC^*_\sigma(\theta))\Big]\Pi_\sigma(\theta),$$

(cf (3.1)), where $\theta\in\mathbb{T}^d\setminus\mathcal{C}_*$,

$$\chi^\pm_{\tau,r_1}(\theta)=(1+\mathrm{sign}(r_1\pm\partial_1\omega_\sigma(\theta)\tau))/2. \tag{5.1}$$

Theorem 5.2 *(see Theorem 2.10 in [17]) Let conditions* **(a)–(d)**, **V1'**, **V2'** *and* **E1–E6** *hold. Then for any* $\tau\neq 0$, $r\in\mathbb{R}^d$ *with* $r_1\ge 0$, *the correlation functions of* $\mu_{\tau/\varepsilon,r/\varepsilon}$ *converge to a limit,*

$$\lim_{\varepsilon\to 0}Q_{\varepsilon,\tau/\varepsilon}([r/\varepsilon]+z,[r/\varepsilon]+z')=Q_{\tau,r}(z,z'). \tag{5.2}$$

Here $z,z'\in\mathbb{Z}^d$ *if* $r_1>0$, $z,z'\in\mathbb{Z}^d_+$ *if* $r_1=0$,

$$Q_{\tau,r}(z,z')=\left\{\begin{array}{ll}\mathbf{q}_{\tau,r}(z-z')=g_{\tau,r}(z-z')+g_{\tau,\tilde r}(\tilde z-\tilde z'), & if\ r_1>0,\\ g_{\tau,r}(z-z')-g_{\tau,r}(z-\tilde z')-g_{\tau,r}(\tilde z-z')+g_{\tau,r}(\tilde z-\tilde z'), & if\ r_1=0,\end{array}\right. \tag{5.3}$$

where $\tilde r:=(-r_1,\bar r)$, $\bar r=(r_2,\dots,r_d)$, *and the matrix* $g_{\tau,r}(z)$ *is defined above.*

Corollary 5.3 *Let* $r\in\mathbb{R}^d_+\equiv\{r\in\mathbb{R}^d: r_1>0\}$ *and* $\tau>0$. *For each* $\sigma=1,\dots,s$, *the* σ*-band of* $\hat{\mathbf{q}}_{\tau,r}(\theta)$, $\theta\in\mathbb{T}^d\setminus\mathcal{C}_*$, *satisfies the following "hydrodynamic" (Euler type) equation:*

$$\partial_\tau f_\sigma(\tau,r;\theta)=iC_\sigma(\theta)\nabla\omega_\sigma(\theta)\cdot\nabla_r f_\sigma(\tau,r;\theta),\quad r\in\mathbb{R}^d_+,\quad \tau>0,$$

with the initial condition (3.7) (if $\tau=0$*) and with the boundary condition (if* $r_1=0$*) expressed by* $\hat{\mathbf{R}}_0$. *In particular, if* $\hat{\mathbf{R}}_0(r,\tilde\theta)=\hat{\mathbf{R}}_0(r,\theta)$, *the boundary condition has a form*

$$f_\sigma|_{r_1=0}=\Pi_\sigma(\theta)\frac{1}{2}\Big(P^\sigma_++C_\sigma(\theta)P^\sigma_+C^*_\sigma(\theta)+iC_\sigma(\theta)P^\sigma_--iP^\sigma_-C^*_\sigma(\theta)\Big)\Pi_\sigma(\theta),\ \ \bar r\in\mathbb{R}^{d-1},\ \ \tau>0,$$

where $P^\sigma_\pm$ *stands for the* $2n\times 2n$ *matrix-valued function,*

$$P^\sigma_\pm=\frac{1}{2}\Big(\hat{\mathbf{R}}_0\left(|\partial_1\omega_\sigma(\theta)|\tau,\bar r+\nabla_{\bar\theta}\omega_\sigma(\theta)\tau,\theta\right)\pm\hat{\mathbf{R}}_0\left(|\partial_1\omega_\sigma(\theta)|\tau,\bar r-\nabla_{\bar\theta}\omega_\sigma(\theta)\tau,\theta\right)\Big).$$

Introduce the complex-valued field (cf. (3.8))

$$a_+(x) = \frac{1}{\sqrt{2}}\Big((\mathcal{V}_+^{1/4}u_0)(x) + i(\mathcal{V}_+^{-1/4}u_1)(x)\Big) \in \mathbb{C}^n\,, \quad x \in \mathbb{Z}^d\,, \tag{5.4}$$

where

$$(\mathcal{V}_+^k u)(x) := \sum_{z\in\mathbb{Z}^d_+} (V^k(x-z) - V^k(x-\tilde z))u(z), \quad \text{with } V^k(z) := F^{-1}_{\theta\to z}\left(\hat V^k(\theta)\right).$$

Let us introduce the scaled $n\times n$ Wigner matrix (cf. (3.9))

$$W_+^\varepsilon(\tau,r;\theta) = \sum_{y\in\mathbb{Z}^d} e^{i\theta\cdot y}\,\mathbb{E}^\varepsilon_{\tau/\varepsilon}\Big(a_+^*([r/\varepsilon+y/2])\otimes a_+([r/\varepsilon-y/2])\Big), \quad r\in\mathbb{R}^d_+,$$

where $a_+(x)$ is given in (5.4). By conditions **V1'** and **V2'**, for fixed $r\in\mathbb{R}^d_+$,

$$\lim_{\varepsilon\to 0} W_+^\varepsilon(0,r;\theta) = W(0,r;\theta),$$

uniformly on $\theta\in\mathbb{T}^d\setminus\mathcal{C}_*$, where $W(0,r;\theta)$ is defined in (3.10).

We also define the limit Wigner matrix as follows (cf (3.11))

$$W_+^p(\tau,r;\theta) = \begin{cases} \sum\limits_{\sigma=1}^s \Pi_\sigma(\theta)W(0,r-\tau\nabla\omega_\sigma(\theta);\theta)\Pi_\sigma(\theta), & \text{if } r_1>\tau\partial_1\omega_\sigma(\theta),\\ \sum\limits_{\sigma=1}^s \Pi_\sigma(\theta)W(0,-r_1+\tau\partial_1\omega_\sigma(\theta),\bar r-\tau\nabla_{\bar\theta}\,\omega_\sigma(\theta);\tilde\theta)\Pi_\sigma(\theta), & \text{if } r_1<\tau\partial_1\omega_\sigma(\theta),\end{cases} \tag{5.5}$$

where $\tilde\theta = (-\theta_1,\bar\theta)$, $\bar\theta = (\theta_2,\ldots,\theta_d)$, $\nabla_{\bar\theta}\,\omega_\sigma(\theta) = (\partial_2\omega_\sigma(\theta),\ldots,\partial_d\omega_\sigma(\theta))$, $\partial_k = \partial/\partial\theta_k$.

Theorem 5.4 *(see Theorem 2.12 in [17]) Let conditions* **V1'**, **V2'** *and* **E1**–**E6** *hold. Then for any $r\in\mathbb{R}^d_+$ and $\tau>0$, the following limit exists in the sense of distributions,*

$$\lim_{\varepsilon\to 0} W_+^\varepsilon(\tau,r;\theta) = W_+^p(\tau,r;\theta)\,. \tag{5.6}$$

Corollary 5.5 *(see Corollary 2.13 in [17]) Denote by $W_\sigma^p(\tau,r;\theta)$, $\sigma=1,\ldots,s$, the σ-th band of the Wigner function $W_+^p(\tau,r;\theta)$. Then, for any fixed $\sigma=1,\ldots,s$, W_σ^p is a solution of the "energy transport" equation*

$$\partial_\tau W_\sigma^p(\tau,r;\theta) + \nabla\omega_\sigma(\theta)\cdot\nabla_r W_\sigma^p(\tau,r;\theta) = 0, \quad \tau>0, \quad r\in\mathbb{R}^d_+,$$

with the initial and boundary conditions

$$\begin{aligned} W_\sigma^p(\tau,r;\theta)|_{\tau=0} &= W_\sigma(0,r;\theta), \quad r\in\mathbb{R}^d_+,\\ W_\sigma^p(\tau,r;\theta)|_{r_1=0} &= b_\sigma(\tau,\bar r;\theta), \quad \bar r\in\mathbb{R}^{d-1}, \quad \tau>0. \end{aligned} \tag{5.7}$$

Here $W_\sigma(0,r;\theta)$ is the σ-th band of the initial Wigner matrix $W(0,r;\theta)$ (see (3.10)),

$$b_\sigma(\tau,\bar r;\theta) := \begin{cases} W_\sigma(0,-\tau\partial_1\omega_\sigma(\theta),\bar r-\tau\nabla_{\bar\theta}\,\omega_\sigma(\theta);\theta), & \text{if } \partial_1\omega_\sigma(\theta)<0,\\ W_\sigma(0,\tau\partial_1\omega_\sigma(\theta),\bar r-\tau\nabla_{\bar\theta}\,\omega_\sigma(\theta);\tilde\theta), & \text{if } \partial_1\omega_\sigma(\theta)>0.\end{cases}$$

In particular, if we assume that $\hat{\mathbf{R}}_0(r,\tilde\theta) = \hat{\mathbf{R}}_0(r,\theta)$ for $r\in\mathbb{R}^d$, $\theta\in\mathbb{T}^d$, then the boundary condition (5.7) can be rewritten in the form

$$W_\sigma^p(\tau,r;\theta)|_{r_1=0} = W_\sigma(0,\tau|\partial_1\omega_\sigma(\theta)|,\bar r-\tau\nabla_{\bar\theta}\,\omega_\sigma(\theta);\theta), \quad \bar r\in\mathbb{R}^{d-1}, \quad \tau>0.$$

Remark (see [17, Theorem 2.15]) Let the measures μ_0^ε satisfy the mixing condition (of the Rosenblatt or Ibragimov type). Then for $\tau \neq 0$, $r \in \mathbb{R}^d$ with $r_1 \geq 0$, in the sense of weak convergence on $\mathcal{H}_{\alpha,+}$, $\lim\limits_{\varepsilon\to 0} \mu^\varepsilon_{\tau/\varepsilon, r/\varepsilon} = \mu^G_{\tau,r}$. The measure $\mu^G_{\tau,r}$ is a Gaussian measure on $\mathcal{H}_{\alpha,+}$, which is invariant under the time translation $U_+(t)$. $\mu^G_{\tau,r}$ has mean zero and covariance $Q_{\tau,r}(z,z')$ defined by (5.3).

5.2 Second approximation

In this subsection we treat the main result of this paper. Let us introduce the matrix $g^\varepsilon_{\tau,r}(z)$, $z \in \mathbb{Z}^d$, $r \in \mathbb{R}^d$, $\tau \neq 0$, $\varepsilon > 0$, which has the following form in the Fourier transform (cf (3.12))

$$\hat{g}^\varepsilon_{\tau,r}(\theta) = \frac{1}{4}\sum_{\sigma=1}^{s} \Pi_\sigma(\theta)\Big[\sum_{\pm}(I \pm iC_\sigma(\theta))\mathbf{A}^\pm_{\varepsilon,\sigma}(\tau, r;\theta)(I \mp iC^*_\sigma(\theta))\Big]\Pi_\sigma(\theta), \quad \theta \in \mathbb{T}^d \setminus \mathcal{C}_*, \quad (5.8)$$

where the matrices $C_\sigma(\theta)$ are defined in (3.4),

$$\mathbf{A}^\pm_{\varepsilon,\sigma}(\tau, r;\theta) = A^\pm_{\varepsilon,\sigma}(\tau, r;\theta)\chi^\pm_{\tau/\varepsilon, r_1}(\theta),$$

with the matrix-valued functions $A^\pm_{\varepsilon,\sigma}(\tau, r;\theta)$ from (3.13) and $\chi^\pm_{\tau,r_1}$ from (5.1).

Theorem 5.6 *Let conditions* **(a)**–**(d')**, **V1'**, **V2'** *and* **E1**–**E6** *hold. Then for any $\tau \neq 0$ and $r \in \mathbb{R}^d$ with $r_1 \geq 0$, the correlation functions of measures $\mu_{\tau/\varepsilon^2, r/\varepsilon}$ have the following asymptotics*

$$\lim_{\varepsilon\to 0}\Big(Q_{\varepsilon,\tau/\varepsilon^2}([r/\varepsilon]+z, [r/\varepsilon]+z') - Q^\varepsilon_{\tau,r}(z,z')\Big) = 0, \quad (5.9)$$

where $z, z' \in \mathbb{Z}^d$ if $r_1 > 0$, $z, z' \in \mathbb{Z}^d_+$ if $r_1 = 0$,

$$Q^\varepsilon_{\tau,r}(z,z') = \begin{cases} \mathbf{q}^\varepsilon_{\tau,r}(z-z') = g^\varepsilon_{\tau,r}(z-z') + g^\varepsilon_{\tau,\tilde{r}}(\tilde{z}-\tilde{z}'), & \text{if } r_1 > 0,\\ g^\varepsilon_{\tau,r}(z-z') - g^\varepsilon_{\tau,r}(z-\tilde{z}') - g^\varepsilon_{\tau,r}(\tilde{z}-z') + g^\varepsilon_{\tau,r}(\tilde{z}-\tilde{z}'), & \text{if } r_1 = 0,\end{cases}$$

$g^\varepsilon_{\tau,r}(z) = F^{-1}_{\theta\to z}[\hat{g}^\varepsilon_{\tau,r}(\theta)]$ and $\hat{g}^\varepsilon_{\tau,r}(\theta)$ defined by (5.8).

Remark. The matrices $\hat{\mathbf{q}}_{\tau,r}(\theta)$ and $\hat{\mathbf{q}}^\varepsilon_{\tau,r}(\theta)$ satisfy the equilibrium condition (see Remarks 3.4 (i)).

Set $\tau = \varepsilon t$. In this case, $\mathbf{A}^\pm_{\varepsilon,\sigma}(\varepsilon t, r;\theta)|_{\varepsilon=0} = \hat{\mathbf{R}}_0(r \pm \nabla\omega_\sigma(\theta)t, \theta)\chi^\pm_{t,r_1}(\theta)$, and, then, $\hat{g}^\varepsilon_{\varepsilon t,r}(\theta)|_{\varepsilon=0} = \hat{g}_{t,r}(\theta)$. Therefore, for $r_1 > 0$, $\hat{\mathbf{q}}^\varepsilon_{\varepsilon t,r}(\theta)|_{\varepsilon=0} = \hat{\mathbf{q}}_{t,r}(\theta)$, where $\hat{\mathbf{q}}_{t,r}(\theta) = F_{z\to\theta}[\mathbf{q}_{t,r}(z)]$ and $\mathbf{q}_{t,r}(z)$ from (5.3).

Corollary 5.7 *Let $r_1 > 0$ and $t > 0$. Then for each $\sigma = 1, \ldots, s$, the matrix-valued function $F_\sigma \equiv F^\varepsilon_\sigma(t, r;\theta) = \Pi_\sigma(\theta)\hat{\mathbf{q}}^\varepsilon_{\varepsilon t,r}(\theta)\Pi_\sigma(\theta)$, $\theta \in \mathbb{T}^d \setminus \mathcal{C}_*$, evolves according to the following mixing problem*

$$\begin{aligned} \partial_t F_\sigma &= iC_\sigma(\theta)\left(\nabla\omega_\sigma(\theta)\cdot\nabla_r F_\sigma + \frac{i\varepsilon}{2}\nabla^2\omega_\sigma(\theta)\cdot\nabla^2_r F_\sigma\right), \quad r_1 > 0, \quad t > 0,\\ F_\sigma|_{t=0} &= \frac{1}{2}\Pi_\sigma(\theta)\left(\hat{\mathbf{R}}_0(r,\theta) + C_\sigma(\theta)\hat{\mathbf{R}}_0(r,\theta)C^*_\sigma(\theta)\right)\Pi_\sigma(\theta), \quad r_1 > 0,\\ F_\sigma|_{r_1=0} &= \frac{1}{4}\sum_{\pm}\Pi_\sigma(\theta)(I \pm iC_\sigma(\theta))\mathbf{A}^\pm_{\varepsilon,\sigma}(\varepsilon t, r;\theta)|_{r_1=0}(I \mp iC^*_\sigma(\theta))\Pi_\sigma(\theta), \quad t > 0.\end{aligned}$$

6 Convergence of correlation functions

6.1 Bounds for initial covariance

Definition 6.1 *By* $\ell^p\equiv\ell^p(\mathbb{Z}^d)\otimes\mathbb{R}^n$ *(by* $\ell^p_+\equiv\ell^p(\mathbb{Z}^d_+)\otimes\mathbb{R}^n$*), where* $p\ge 1$ *and* $n\ge 1$, *denote the space of sequences* $f(z)=(f_1(z),\ldots,f_n(z))$ *endowed with norm* $\|f\|_{\ell^p}=\Big(\sum_{z\in\mathbb{Z}^d}|f(z)|^p\Big)^{1/p}$, *respectively,* $\|f\|_{\ell^p_+}=\Big(\sum_{z\in\mathbb{Z}^d_+}|f(z)|^p\Big)^{1/p}$.

The following lemma follows from condition **V2**.

Lemma 6.2 *Let condition* **V2** *hold. Then, for* $i,j=0,1$, *the following bounds hold:*

$$\sum_{z'\in\mathbb{Z}^d_+}|Q^{ij}_\varepsilon(z,z')| \ \le\ C<\infty \quad \text{for all } z\in\mathbb{Z}^d_+,$$
$$\sum_{z\in\mathbb{Z}^d_+}|Q^{ij}_\varepsilon(z,z')| \ \le\ C<\infty \quad \text{for all } z'\in\mathbb{Z}^d_+.$$

Here the constant C *does not depend on* $z,z'\in\mathbb{Z}^d_+$ *and* $\varepsilon>0$.

Corollary 6.3 *By the Shur lemma, it follows from Lemma 6.2 that*

$$|\langle Q_\varepsilon(z,z'),\Phi(z)\otimes\Psi(z')\rangle_+|\le C\|\Phi\|_{\ell^2_+}\|\Psi\|_{\ell^2_+}, \quad \text{for any } \Phi,\Psi\in\ell^2_+,$$

where the constant C *does not depend on* $\varepsilon>0$.

6.2 Stationary phase method

By (2.8) and (2.9) we see that $\hat{\mathcal{G}}_t(\theta)$ is of the form

$$\hat{\mathcal{G}}_t(\theta)=\begin{pmatrix}\cos\Omega t & \sin\Omega t\ \Omega^{-1}\\ -\sin\Omega t\ \Omega & \cos\Omega t\end{pmatrix}, \tag{6.1}$$

where $\Omega=\Omega(\theta)$ is the Hermitian matrix defined by (2.4). Hence, by Lemma 2.2 (iii), by formulas $\cos\omega_\sigma(\theta)t=(e^{i\omega_\sigma t}+e^{-i\omega_\sigma t})/2$, $\sin\omega_\sigma(\theta)t=(e^{i\omega_\sigma t}-e^{-i\omega_\sigma t})/(2i)$ and by (3.4), the matrix $\mathcal{G}_t(x)$ can be rewritten in the form

$$\mathcal{G}_t(x)=\sum_{\pm,\sigma=1}^{s}\int\limits_{\mathbb{T}^d}e^{-ix\cdot\theta}e^{\pm i\omega_\sigma(\theta)\,t}c^{\mp}_\sigma(\theta)\,d\theta, \tag{6.2}$$

where $c^{\mp}_\sigma(\theta):=\Pi_\sigma(\theta)(I\mp iC_\sigma(\theta))/2$. We are going to apply the stationary phase arguments to the integral (6.2) which require a smoothness in θ. Then we have to choose certain smooth branches of the functions $c^{\pm}_\sigma(\theta)$ and $\omega_\sigma(\theta)$ and cut off all singularities. First, introduce the *critical set* as

$$\mathcal{C}=\mathcal{C}_0\bigcup\mathcal{C}_*\bigcup_{\sigma=1}^{s}\Big(\mathcal{C}_\sigma\bigcup_{i=1}^{d}\Big\{\theta\in\mathbb{T}^d\setminus\mathcal{C}_*:\ \frac{\partial^2\omega_\sigma(\theta)}{\partial\theta_i^2}=0\Big\}\bigcup\Big\{\theta\in\mathbb{T}^d\setminus\mathcal{C}_*:\ \frac{\partial\,\omega_\sigma(\theta)}{\partial\theta_1}=0\Big\}\Big), \tag{6.3}$$

with $\mathcal{C}_*$ as in Lemma 2.2 and sets $\mathcal{C}_0$ and $\mathcal{C}_\sigma$ defined by (2.5). Obviously, $\mathrm{mes}\,\mathcal{C}=0$ (see [11, Lemmas 2.2, 2.3]). Secondly, fix an $\delta>0$ and choose a finite partition of unity,

$$f(\theta)+g(\theta)=1,\quad g(\theta)=\sum_{k=1}^{K} g_k(\theta),\quad \theta\in\mathbb{T}^d, \tag{6.4}$$

where f, g_k are non-negative functions in $C_0^\infty(\mathbb{T}^d)$, and

$$\mathrm{supp}\, f\subset\{\theta\in\mathbb{T}^d:\ \mathrm{dist}(\theta,\mathcal{C})<\delta\},\ \ \mathrm{supp}\, g_k\subset\{\theta\in\mathbb{T}^d:\ \mathrm{dist}(\theta,\mathcal{C})\geq\delta/2\}. \tag{6.5}$$

Then we represent $\mathcal{G}_t(x)$ in the form $\mathcal{G}_t(x)=\mathcal{G}_t^f(x)+\mathcal{G}_t^g(x)$, where

$$\mathcal{G}_t^f(x)=F^{-1}_{\theta\to x}[f(\theta)\,\hat{\mathcal{G}}_t(\theta)],\quad \mathcal{G}_t^g(x)=F^{-1}_{\theta\to x}[g(\theta)\,\hat{\mathcal{G}}_t(\theta)]. \tag{6.6}$$

By Lemma 2.2 and the compactness arguments, we can choose the supports of g_k so small that the eigenvalues $\omega_\sigma(\theta)$ and the amplitudes $c^\pm_\sigma(\theta)$ are real-analytic functions inside the $\mathrm{supp}\, g_k$ for every k. (We do not label the functions by the index k to simplify the notation.) The Parseval identity, (6.1), and condition **E6** imply

$$\begin{aligned}\|\mathcal{G}_t^f(\cdot)\|^2_{\ell^2} &= C\int\limits_{\mathbb{T}^d}|\hat{\mathcal{G}}_t(\theta)|^2|f(\theta)|^2\,d\theta\leq C\int\limits_{\mathrm{dist}(\theta,\mathcal{C})<\delta}|\hat{\mathcal{G}}_t(\theta)|^2\,d\theta\to 0\ \text{ as }\ \delta\to 0,\\ \|\mathcal{G}_t^g(\cdot)\|^2_{\ell^2} &\leq C\int\limits_{\mathbb{T}^d}|\hat{\mathcal{G}}_t(\theta)|^2\,d\theta\leq C_1<\infty,\end{aligned} \tag{6.7}$$

uniformly in $t\in\mathbb{R}$. For the function $\mathcal{G}_t^g(x)$, the following lemma holds.

Lemma 6.4 *(see [14, Lemma 4.2]) Let conditions* **E1**–**E4** *and* **E6** *hold. Then the following bounds hold.*

(i) $\sup_{x\in\mathbb{Z}^d}|\mathcal{G}_t^g(x)|\leq C\ t^{-d/2}$.

(ii) For any $p>0$, *there exist* $C_p,\gamma_g>0$ *such that* $|\mathcal{G}_t^g(x)|\leq C_p(|t|+|x|+1)^{-p}$ *for* $|x|\geq\gamma_g t$.

6.3 Proof of Theorem 5.6

The representation (4.6) yields

$$Q_{\varepsilon,t}(z,z')=\mathbb{E}_0^\varepsilon(Y(z,t)\otimes Y(z',t))=\sum_{x,y\in\mathbb{Z}^d_+}\mathcal{G}_{t,+}(z,x)Q_\varepsilon(x,y)\mathcal{G}_{t,+}(z',y)^T,\ \ z,z'\in\mathbb{Z}^d_+,$$

for any $t\in\mathbb{R}$. It follows from condition (4.5) and from formulas (2.8) and (2.9) that $\mathcal{G}_t(z)=\mathcal{G}_t(\tilde z)$ with $\tilde z=(-z_1,z_2,\ldots,z_d)$. In this case, by (4.7), the covariance $Q_{\varepsilon,t}(z,z')$ can be decomposed into the sum of fourth terms,

$$Q_{\varepsilon,t}(z,z')=S_{\varepsilon,t}(z,z')-S_{\varepsilon,t}(\tilde z,z')-S_{\varepsilon,t}(z,\tilde z')+S_{\varepsilon,t}(\tilde z,\tilde z'),\quad z,z'\in\mathbb{Z}^d_+,$$

where

$$S_{\varepsilon,t}(z,z'):=\sum_{x,y\in\mathbb{Z}^d_+}\mathcal{G}_t(z-x)Q_\varepsilon(x,y)\mathcal{G}_t(z'-y)^T.$$

Proposition 6.5 *Let* $r \in \mathbb{R}^d$ *and* $z, z' \in \mathbb{Z}^d$. *Then*

$$\lim_{\varepsilon\to+0}(S_{\varepsilon,\tau/\varepsilon^2}([r/\varepsilon]+z,[r/\varepsilon]+z') - g^\varepsilon_{\tau,r}(z-z')) = 0, \tag{6.8}$$

where $g^\varepsilon_{\tau,r}(z)$ *is defined by (5.8).*

This proposition implies Theorem 5.6. Indeed, let $r_1 = 0$. Then $\tilde r/\varepsilon = r/\varepsilon$, and for $z, z' \in \mathbb{Z}^d_+$,

$$\begin{aligned} Q_{\varepsilon,t}([r/\varepsilon]+z,[r/\varepsilon]+z') &= S_{\varepsilon,t}([r/\varepsilon]+z,[r/\varepsilon]+z') - S_{\varepsilon,t}([r/\varepsilon]+\tilde z,[r/\varepsilon]+z') \\ &\quad - S_{\varepsilon,t}([r/\varepsilon]+z,[r/\varepsilon]+\tilde z') + S_{\varepsilon,t}([r/\varepsilon]+\tilde z,[r/\varepsilon]+\tilde z'). \end{aligned}$$

Therefore, convergence (6.8) implies (5.9).

Let $r_1 > 0$. In this case, the matrix-valued functions $S_{\varepsilon,\tau/\varepsilon^2}([\tilde r/\varepsilon]+\tilde z,[r/\varepsilon]+z')$ and $S_{\varepsilon,\tau/\varepsilon^2}([r/\varepsilon]+z,[\tilde r/\varepsilon]+\tilde z')$ vanish as $\varepsilon\to+0$, and

$$S_{\varepsilon,\tau/\varepsilon^2}([\tilde r/\varepsilon]+\tilde z,[\tilde r/\varepsilon]+\tilde z') - g^\varepsilon_{\tau,\tilde r}(\tilde z-\tilde z') \to 0, \quad \varepsilon\to+0.$$

It can be proved similarly to Proposition 6.5.

Proof of Proposition 6.5. Let us denote

$$\bar Q_\varepsilon(x,y) = \begin{cases} Q_\varepsilon(x,y) & \text{for } x,y\in\mathbb{Z}^d_+, \\ 0 & \text{otherwise} \end{cases}$$

The partition (6.4), Corollary 6.3 and the bound (6.7) yield

$$S_{\varepsilon,t}(z,z') = \sum_{x,y\in\mathbb{Z}^d} \mathcal{G}^g_t(z-x)\bar Q_\varepsilon(x,y)\mathcal{G}^g_t(z'-y)^T + o(1),$$

where $\mathcal{G}^g_t$ is defined in (6.6), $o(1)\to 0$ as $\delta\to 0$ uniformly in $t\in\mathbb{R}$ and $z,z'\in\mathbb{Z}^d$. In particular, setting $t=\tau/\varepsilon^2$, $z=[r/\varepsilon]+l$ and $z'=[r/\varepsilon]+p$ we obtain

$$\begin{aligned} S_{\varepsilon,\tau/\varepsilon^2}([r/\varepsilon]+l,[r/\varepsilon]+p) &= \sum_{x,y\in\mathbb{Z}^d} \mathcal{G}^g_{\tau/\varepsilon^2}([r/\varepsilon]+l-x)\bar Q_\varepsilon(x,y)\mathcal{G}^g_{\tau/\varepsilon^2}([r/\varepsilon]+p-y)^T + o(1) \\ &= \sum_{x,y\in\mathbb{Z}^d} \mathcal{G}^g_{\tau/\varepsilon^2}(l+x)\bar Q_\varepsilon([r/\varepsilon]-x,[r/\varepsilon]-y)\mathcal{G}^g_{\tau/\varepsilon^2}(p+y)^T + o(1). \end{aligned} \tag{6.9}$$

Let $c=\gamma_g+\max(|l|,|p|)$ with γ_g from Lemma 6.4. Then Lemma 6.4 (ii) and condition **V2'** imply that the series in (6.9) can be taken over $x,y\in[-c\tau/\varepsilon^2, c\tau/\varepsilon^2]^d$.

By definition, the function $\bar R$ is equal to

$$\bar R(r,x,y) = \begin{cases} R(r,x,y) & \text{if } x,y\in\mathbb{Z}^d_+, \\ 0 & \text{otherwise} \end{cases}$$

The asymptotics of $S_{\varepsilon,\tau/\varepsilon^2}$ is not changed when we replace $\bar Q_\varepsilon([r/\varepsilon]-x,[r/\varepsilon]-y)$ in the r.h.s. of (6.9) by $\bar R(\ldots) \equiv \bar R(\varepsilon[r/\varepsilon]-\varepsilon x,[r/\varepsilon]-x,[r/\varepsilon]-y)$, i.e.,

$$S_{\varepsilon,\tau/\varepsilon^2}([r/\varepsilon]+l,[r/\varepsilon]+p) = \sum_{x,y\in[-c\tau/\varepsilon^2,c\tau/\varepsilon^2]^d} \mathcal{G}^g_{\tau/\varepsilon^2}(l+x)\bar R(\ldots)\mathcal{G}^g_{\tau/\varepsilon^2}(p+y)^T + o(1). \tag{6.10}$$

Indeed, by Lemma 6.4 (i) and condition (4.12),

$$\begin{aligned}
&\Big|\sum_{x,y\in[-c\tau/\varepsilon^2,c\tau/\varepsilon^2]^d}\mathcal{G}^g_{\tau/\varepsilon^2}(l+x)\Big(\bar{Q}_\varepsilon([r/\varepsilon]-x,[r/\varepsilon]-y)-\bar{R}(\ldots)\Big)\mathcal{G}^g_{\tau/\varepsilon^2}(p+y)^T\Big|\\
\le\ & C\sum_{y\in\mathbb{Z}^d}\min[(1+|x-y|)^{-\gamma},\varepsilon b|x-y|],
\end{aligned}$$

where the last series is order of $\varepsilon^{(\gamma-d)/(\gamma+1)}$. This goes to zero as $\varepsilon\to 0$, since $\gamma>d$. Using Lemma 6.4 and properties of R, we can take the series in (6.10) over $x,y\in\mathbb{Z}^d$. Hence,

$$S_{\varepsilon,\tau/\varepsilon^2}([r/\varepsilon]+l,[r/\varepsilon]+p)=\sum_{x,y\in\mathbb{Z}^d}\mathcal{G}^g_{\tau/\varepsilon^2}(l+x)\bar{R}(\ldots)\mathcal{G}^g_{\tau/\varepsilon^2}(p+y)^T+o(1),\ \ \varepsilon\to 0. \quad (6.11)$$

Let us split the function $\bar{R}$ into the following three matrix functions:

$$R^+(r,x,y)\ :=\ \frac{1}{2}\mathbf{R}_0(r,x-y), \quad (6.12)$$

$$R^-(r,x,y)\ :=\ \frac{1}{2}\mathbf{R}_0(r,x-y)\,\mathrm{sign}(y_1), \quad (6.13)$$

$$R^0(r,x,y)\ :=\ \bar{R}(r,x,y)-R^+(r,x,y)-R^-(r,x,y). \quad (6.14)$$

Next, introduce the matrices

$$\begin{aligned}
S^a_{\varepsilon,\tau/\varepsilon^2}\ &\equiv\ S^a_{\varepsilon,\tau/\varepsilon^2}([r/\varepsilon]+l,[r/\varepsilon]+p)\\
&=\ \sum_{x,y\in\mathbb{Z}^d}\mathcal{G}^g_{\tau/\varepsilon^2}(l+x)\bar{R}^a(\varepsilon[r/\varepsilon]-\varepsilon x,[r/\varepsilon]-x,[r/\varepsilon]-y)\mathcal{G}^g_{\tau/\varepsilon^2}(p+y)^T, \quad (6.15)
\end{aligned}$$

for each $a=\{+,-,0\}$ and split $S_{\varepsilon,\tau/\varepsilon^2}$ into three terms, $S_{\varepsilon,\tau/\varepsilon^2}=S^+_{\varepsilon,\tau/\varepsilon^2}+S^-_{\varepsilon,\tau/\varepsilon^2}+S^0_{\varepsilon,\tau/\varepsilon^2}$. The convergence (6.8) results now from the following three Lemmas 6.6–6.8, since (see (5.8)) $g^\varepsilon_{\tau,r}(z)=(1/2)q^\varepsilon_{\tau,r}(z)+(1/2)f^\varepsilon_{\tau,r}(z)$, where $q^\varepsilon_{\tau,r}(z)$ is defined by (3.12) and $f^\varepsilon_{\tau,r}(z)$ is defined in Lemma 6.7.

Lemma 6.6 $\lim\limits_{\varepsilon\to 0}(S^+_{\varepsilon,\tau/\varepsilon^2}([r/\varepsilon]+l,[r/\varepsilon]+p)-(1/2)q^\varepsilon_{\tau,r}(l-p))=0$, $l,p\in\mathbb{Z}^d$, *where* $q^\varepsilon_{\tau,r}(l)$ *is defined by (3.12).*

Lemma 6.7 $\lim\limits_{\varepsilon\to 0}(S^-_{\varepsilon,\tau/\varepsilon^2}([r/\varepsilon]+l,[r/\varepsilon]+p)-(1/2)f^\varepsilon_{\tau,r}(l-p))=0$, $l,p\in\mathbb{Z}^d$.
Here $f^\varepsilon_{\tau,r}(l)$ *stands for the matrix which is defined similarly to* $q^\varepsilon_{\tau,r}(l)$ *but with the matrix* $A^\pm_{\varepsilon,\sigma}(\tau,r;\theta)\,\mathrm{sign}(r_1\pm\partial_1\omega_\sigma(\theta)\tau/\varepsilon)$ *instead of* $A^\pm_{\varepsilon,\sigma}(\tau,r;\theta)$ *in the r.h.s. of (3.12).*

Lemma 6.8 $\lim\limits_{\varepsilon\to 0}S^0_{\varepsilon,\tau/\varepsilon^2}([r/\varepsilon]+l,[r/\varepsilon]+p)=0$, $l,p\in\mathbb{Z}^d$.

The proofs of Lemmas 6.7 and 6.8 see in Appendices A and B, resp.

Proof of Lemma 6.6. *Step (i)*: By (6.12) and (6.15), the function $S^+_{\varepsilon,\tau/\varepsilon^2}$ can be represented as

$$S^+_{\varepsilon,\tau/\varepsilon^2}=\frac{1}{2}\sum_{x\in\mathbb{Z}^d}\mathcal{G}^g_{\tau/\varepsilon^2}(l+x)\sum_{y\in\mathbb{Z}^d}\mathbf{R}_0(\varepsilon[r/\varepsilon]-\varepsilon x,y-x)\mathcal{G}^g_{\tau/\varepsilon^2}(p+y)^T.$$

Using the Fourier transform and the Parseval identity we rewrite $S^+_{\varepsilon,\tau/\varepsilon^2}$ as

$$S^+_{\varepsilon,\tau/\varepsilon^2}=(2\pi)^{-2d}\frac{1}{2}\sum_{x\in\mathbb{Z}^d}\int_{\mathbb{T}^{2d}} e^{-il\cdot\theta'+ip\cdot\theta+ix\cdot(\theta-\theta')}\hat{\mathcal{G}}^g_{\tau/\varepsilon^2}(\theta')\hat{\mathbf{R}}_0(\varepsilon[r/\varepsilon]-\varepsilon x,\theta)\hat{\mathcal{G}}^g_{\tau/\varepsilon^2}(\theta)^*\,d\theta d\theta'. \quad (6.16)$$

Change variables in (6.16): $(\theta,\theta')\to(\theta,\varphi)$, $\varphi=\theta-\theta'$ and rewrite $S^+_{\varepsilon,\tau/\varepsilon^2}$ in the form

$$S^+_{\varepsilon,\tau/\varepsilon^2}=(2\pi)^{-d}\frac{1}{2}\int_{\mathbb{T}^d} e^{-i(l-p)\cdot\theta}I_\varepsilon(\theta)\hat{\mathcal{G}}^g_{\tau/\varepsilon^2}(\theta)^*\,d\theta, \quad (6.17)$$

where $I_\varepsilon(\theta)$ stands for the matrix-valued function,

$$I_\varepsilon(\theta)=\sum_{x\in\mathbb{Z}^d}(2\pi)^{-d}\int_{\mathbb{T}^d} e^{i(l+x)\cdot\varphi}\hat{\mathcal{G}}^g_{\tau/\varepsilon^2}(\theta-\varphi)\,d\varphi\,\hat{\mathbf{R}}_0(\varepsilon[r/\varepsilon]-\varepsilon x,\theta). \quad (6.18)$$

Step (ii): For simplicity of the proof, let us assume that $\hat{\mathbf{R}}_0(r,\theta)$ satisfies condition (2.11). Under the more weakened condition (2.14) the proof is given in Appendix C.

We apply the Poisson summation formula (see, for example, [24]) and obtain

$$\sum_{x\in\mathbb{Z}^d} e^{ix\cdot\varphi}\hat{\mathbf{R}}_0(\varepsilon[r/\varepsilon]-\varepsilon x,\theta)=\varepsilon^{-d}\sum_{n\in\mathbb{Z}^d}\tilde{\mathbf{R}}_0(-\varphi/\varepsilon-2\pi n/\varepsilon,\theta)e^{i[r/\varepsilon]\cdot\varphi}, \quad (6.19)$$

where $\tilde{\mathbf{R}}_0(\cdot,\theta)$ stands for the Fourier transform of $\hat{\mathbf{R}}_0(\cdot,\theta)$ (see (2.12)). We substitute the last expression to (6.18) and obtain

$$I_\varepsilon(\theta)=(2\pi\varepsilon)^{-d}\int_{\mathbb{T}^d} e^{i(l+[r/\varepsilon])\cdot\varphi}\hat{\mathcal{G}}^g_{\tau/\varepsilon^2}(\theta-\varphi)\Big(\sum_{n\in\mathbb{Z}^d}\tilde{\mathbf{R}}_0(-\varphi/\varepsilon-2\pi n/\varepsilon,\theta)\Big)\,d\varphi+o(1). \quad (6.20)$$

Condition **I4’** implies that

$$\varepsilon^{-d}\sum_{n\neq 0}\tilde{\mathbf{R}}_0(-\varphi/\varepsilon-2\pi n/\varepsilon,\theta)\to 0\quad\text{as }\varepsilon\to 0 \quad (6.21)$$

uniformly in $\theta\in\mathbb{T}^d$, $\varphi\in[-\pi,\pi]^d$. Hence (6.20) and (6.21) yield

$$I_\varepsilon(\theta)=(2\pi\varepsilon)^{-d}\int_{[-\pi,\pi]^d} e^{i\varphi\cdot(l+[r/\varepsilon])}\hat{\mathcal{G}}^g_{\tau/\varepsilon^2}(\theta-\varphi)\tilde{\mathbf{R}}_0(-\varphi/\varepsilon,\theta)\,d\varphi+o(1).$$

Step (iii): Change variables $\varphi\to-\varepsilon\varphi$ in the last integral and obtain

$$I_\varepsilon(\theta)=(2\pi)^{-d}\int_{[-\pi/\varepsilon,\pi/\varepsilon]^d} e^{-i\varepsilon\varphi\cdot(l+[r/\varepsilon])}\hat{\mathcal{G}}^g_{\tau/\varepsilon^2}(\theta+\varepsilon\varphi)\tilde{\mathbf{R}}_0(\varphi,\theta)\,d\varphi+o(1).$$

Now we use the following representation for $\hat{\mathcal{G}}^g_t(\theta)$ (see (6.2)):

$$\hat{\mathcal{G}}^g_t(\theta)=\sum_{\sigma=1}^{s}\sum_{\pm} e^{\pm i\omega_\sigma(\theta)t}\,h^\mp_\sigma(\theta),\quad\text{where } h^\mp_\sigma(\theta)=g(\theta)c^\mp_\sigma(\theta)=g(\theta)\frac{I\mp iC_\sigma(\theta)}{2}\Pi_\sigma(\theta), \quad (6.22)$$

and rewrite $I_\varepsilon(\theta)$ in the form

$$\begin{aligned} I_\varepsilon(\theta) &= (2\pi)^{-d}\sum_{\sigma,\pm}\int\limits_{[-\pi/\varepsilon,\pi/\varepsilon]^d} e^{-i\varepsilon\varphi\cdot(l+[r/\varepsilon])}e^{\pm i\omega_\sigma(\theta+\varepsilon\varphi)\tau/\varepsilon^2}h^\mp_\sigma(\theta+\varepsilon\varphi)\tilde{\mathbf{R}}_0(\varphi,\theta)\,d\varphi+o(1)\\ &= (2\pi)^{-d}\sum_{\sigma,\pm}h^\mp_\sigma(\theta)\int\limits_{[-\pi/\varepsilon,\pi/\varepsilon]^d} e^{-i\varphi\cdot r}e^{\pm i\omega_\sigma(\theta+\varepsilon\varphi)\tau/\varepsilon^2}\tilde{\mathbf{R}}_0(\varphi,\theta)\,d\varphi+o(1), \end{aligned}$$

since $e^{-i\varepsilon\varphi\cdot l}-1=O(\varepsilon)$ and $e^{-i\varphi\cdot\varepsilon[r/\varepsilon]}-e^{-i\varphi\cdot r}=O(\varepsilon)$ by (2.11).

Step (iv): Let us take the Taylor sum representation for $\omega_\sigma(\theta+\varepsilon\varphi)$:

$$\omega_\sigma(\theta+\varepsilon\varphi)\tau/\varepsilon^2=\omega_\sigma(\theta)\tau/\varepsilon^2+\varphi\cdot\nabla\omega_\sigma(\theta)\tau/\varepsilon+(\tau/2)\varphi\cdot H_\sigma(\theta)\varphi+O(\varepsilon|\varphi|^3), \qquad (6.23)$$

where $H_\sigma(\theta)$ denotes the matrix $\nabla^2\omega_\sigma(\theta)$. Hence, by (2.11),

$$I_\varepsilon(\theta)=\sum_{\sigma,\pm}h^\mp_\sigma(\theta)e^{\pm i\omega_\sigma(\theta)\tau/\varepsilon^2}A^\mp_{\varepsilon,\sigma}(\tau,r;\theta)+o(1),\qquad \varepsilon\to 0. \qquad (6.24)$$

Here (see formulas (3.13) and (3.14))

$$\begin{aligned} A^\mp_{\varepsilon,\sigma}(\tau,r;\theta) &= (2\pi)^{-d}\int\limits_{\mathbb{R}^d} e^{-i\varphi\cdot(r\mp\nabla\omega_\sigma(\theta)\tau/\varepsilon)}e^{\pm i(\tau/2)\varphi\cdot H_\sigma(\theta)\varphi}\tilde{\mathbf{R}}_0(\varphi,\theta)\,d\varphi\\ &= F^{-1}_{\varphi\to(r\mp\nabla\omega_\sigma(\theta)\tau/\varepsilon)}\Big[\tilde{\mathbf{R}}_0(\varphi,\theta)e^{\pm i(\tau/2)\varphi\cdot H_\sigma(\theta)\varphi}\Big]\\ &= \int\limits_{\mathbb{R}^d}\hat{\mathbf{R}}_0(r\mp\nabla\omega_\sigma(\theta)\tau/\varepsilon-x,\theta)K^\mp_\sigma(\tau,x,\theta)\,dx, \end{aligned}$$

and $K^\mp_\sigma(\tau,x,\theta)$ ($\tau>0$, $x\in\mathbb{R}^d$, $\theta\in\mathbb{T}^d\setminus\mathcal{C}$) stands for the following matrix-valued function

$$K^\mp_\sigma(\tau,x,\theta)=F^{-1}_{\varphi\to x}[e^{\pm i(\tau/2)\varphi\cdot H_\sigma(\theta)\varphi}]=\frac{e^{\pm i\pi s/4}}{(2\pi\,\tau)^{d/2}}\frac{e^{\mp i/(2\tau)x\cdot H^{-1}_\sigma(\theta)x}}{\sqrt{|\det H_\sigma(\theta)|}}, \qquad (6.25)$$

where s denotes the signature of the matrix $H_\sigma(\theta)$. By definition, the signature of the non-degenerate symmetric matrix A means the signature of the quadratic form with this matrix (or the difference between the sums of positive and negative eigenvalues of A). Formula (6.25) follows from the following equality (see, for example, *Ramanujan's integrals* in [24, section 7.5] or [25, §3]): for any $a>0$,

$$\int\limits_{-\infty}^{+\infty}e^{-ix\varphi\pm ia\varphi^2}\,d\varphi=\sqrt{\frac{\pi}{a}}e^{\pm i\pi/4}e^{\mp ix^2/(4a)},\quad x\in\mathbb{R}^1.$$

Step (v): We substitute (6.24) in (6.17), apply the decomposition (6.22) to $\hat{\mathcal{G}}^g_{\tau/\varepsilon^2}(\theta)^*$ and obtain

$$\begin{aligned} S^+_{\varepsilon,\tau/\varepsilon^2} &= (2\pi)^{-d}\frac{1}{2}\int\limits_{\mathbb{T}^d}e^{-i(l-p)\cdot\theta}\Big(\sum_{\sigma,\pm}h^\mp_\sigma(\theta)e^{\pm i\omega_\sigma(\theta)\tau/\varepsilon^2}A^\mp_{\varepsilon,\sigma}(\tau,r;\theta)\Big)\\ &\qquad\times\Big(\sum_{\sigma',\pm}e^{\pm i\omega_{\sigma'}(\theta)\tau/\varepsilon^2}h^\mp_{\sigma'}(\theta)^*\Big)\,d\theta+o(1),\quad\varepsilon\to 0. \end{aligned} \qquad (6.26)$$

Therefore, to find the asymptotics of $S^+_{\varepsilon,\tau/\varepsilon^2}$ it suffices to study the behaviour as $\varepsilon\to 0$ of the following integrals

$$I^{\pm}_{\sigma\sigma'}(\varepsilon)\equiv(2\pi)^{-d}\frac{1}{2}\int\limits_{\mathbb{T}^d} e^{-i(l-p)\cdot\theta}e^{i(\omega_\sigma(\theta)\pm\omega_{\sigma'}(\theta))\tau/\varepsilon^2}h^-_\sigma(\theta)A^-_{\varepsilon,\sigma}(\tau,r;\theta)h^{\mp}_{\sigma'}(\theta)^*\,d\theta, \tag{6.27}$$

$\sigma,\sigma'=1,\dots,s$. Note that $\sup\limits_{\theta\in\mathbb{T}^d\setminus\mathcal{C}}|A^-_{\varepsilon,\sigma}(\tau,r;\theta)|\le C<\infty$ by condition **I4'**. Hence, the function $h^-_\sigma(\theta)A^-_{\varepsilon,\sigma}(\tau,r;\theta)h^{\mp}_{\sigma'}(\theta)^*\in L^1(\mathbb{T}^d)$ by condition **E6**. Therefore, the oscillatory integrals with $\omega_\sigma(\theta)\pm\omega_{\sigma'}(\theta)\not\equiv\mathrm{const}_\pm$ vanish as $\varepsilon\to 0$ by the Lebesgue–Riemann theorem. Furthermore, the identities $\omega_\sigma(\theta)\pm\omega_{\sigma'}(\theta)\equiv\mathrm{const}_\pm$ in the exponent of (6.27) with $\mathrm{const}_\pm\ne 0$ are impossible by condition **E5**. Hence, only the integrals with $\omega_\sigma(\theta)-\omega_{\sigma'}(\theta)\equiv 0$ contribute to the integral (6.27) since $\omega_\sigma(\theta)+\omega_{\sigma'}(\theta)\equiv 0$ would imply $\omega_\sigma(\theta)\equiv\omega_{\sigma'}(\theta)\equiv 0$ which is impossible by **E4**. Therefore, if $\sigma\ne\sigma'$, $I^{\pm}_{\sigma\sigma'}(\varepsilon)=o(1)$ as $\varepsilon\to 0$. If $\sigma=\sigma'$, $I^+_{\sigma\sigma}(\varepsilon)=o(1)$,

$$I^-_{\sigma\sigma}(\varepsilon)=(2\pi)^{-d}\frac{1}{2}\int\limits_{\mathbb{T}^d} e^{-i(l-p)\cdot\theta}h^-_\sigma(\theta)A^-_{\varepsilon,\sigma}(\tau,r;\theta)h^+_\sigma(\theta)^*\,d\theta+o(1),\quad \varepsilon\to 0.$$

Finally, using (6.26) and the equalities $h^{\mp}_\sigma(\theta)=g(\theta)\Pi_\sigma(\theta)(I\mp iC_\sigma(\theta))/2$ and (3.12), we obtain

$$\begin{aligned}S^+_{\varepsilon,\tau/\varepsilon^2} &= (2\pi)^{-d}\frac{1}{2}\int\limits_{\mathbb{T}^d} e^{-i(l-p)\cdot\theta}\sum_{\sigma=1}^{s}\sum_{\pm}h^{\mp}_\sigma(\theta)A^{\mp}_{\varepsilon,\sigma}(\tau,r;\theta)h^{\pm}_\sigma(\theta)^*\,d\theta+o(1) \qquad (6.28)\\ &= \frac{1}{2}q^\varepsilon_{\tau,r}(l-p)+o(1),\quad \varepsilon\to 0.\quad\blacksquare\end{aligned}$$

7 Convergence of Wigner matrices

Here we prove Theorem 5.4. Theorem 5.6 implies that for any fixed $r\in\mathbb{R}^d_+$, $\tau\ne 0$, and $y\in(2\mathbb{Z})^d$, the following convergence holds,

$$\lim_{\varepsilon\to 0}\mathbb{E}^\varepsilon_{\tau/\varepsilon}(a^*_+([r/\varepsilon]+y/2)\otimes a_+([r/\varepsilon]-y/2))=\mathcal{W}^p_+(\tau,r;y), \tag{7.1}$$

where in the Fourier space one has

$$\begin{aligned}\hat{\mathcal{W}}^p_+(\tau,r;\theta) &= \frac{1}{2}\Big(\Omega^{1/2}\hat{\mathbf{q}}^{00}_{\tau,r}(\theta)\Omega^{1/2}+\Omega^{-1/2}\hat{\mathbf{q}}^{11}_{\tau,r}(\theta)\Omega^{-1/2}\\ &\quad +i\Omega^{1/2}\hat{\mathbf{q}}^{01}_{\tau,r}(\theta)\Omega^{-1/2}-i\Omega^{-1/2}\hat{\mathbf{q}}^{10}_{\tau,r}(\theta)\Omega^{1/2}\Big)=W^p_+(\tau,r;\theta),\ \ r\in\mathbb{R}^d_+, \qquad (7.2)\end{aligned}$$

by formulas (5.5) and (5.3). Therefore, convergence (5.6) follows from (7.1), (7.2) and the following bound:

$$\sup_{\varepsilon>0}\sup_{t\in\mathbb{R}}\sup_{z,z'\in\mathbb{Z}^d}\left|\mathbb{E}^\varepsilon_t(a^*_+(z)\otimes a_+(z'))\right|\le C<\infty.$$

The proof of this bound follows from Lemma 7.1.

Lemma 7.1 *Let conditions* **V2'** *and* **E1**–**E3**, **E6** *hold and let* $\alpha < -d/2$. *Then the following bound holds:* $\sup\limits_{\varepsilon>0}\sup\limits_{t\in\mathbb{R}}\sup\limits_{z,z'\in\mathbb{Z}^d_+}\|Q_{\varepsilon,t}(z,z')\| \le C < \infty$.

Proof. The representation (4.6) gives

$$
\begin{aligned}
Q^{ij}_{\varepsilon,t}(z,z') &= \mathbb{E}^\varepsilon_0\Big(Y^i(z,t)\otimes Y^j(z',t)\Big) = \sum_{y,y'\in\mathbb{Z}^d_+}\sum_{k,l=0,1}\mathcal{G}^{ik}_{t,+}(z,y)Q^{kl}_\varepsilon(y,y')\mathcal{G}^{jl}_{t,+}(z',y') \\
&= \langle Q_\varepsilon(y,y'),\Phi^i_z(y,t)\otimes\Phi^j_{z'}(y',t)\rangle_+,
\end{aligned}
$$

where $\Phi^i_z(y,t)$ is given by

$$
\begin{aligned}
\Phi^i_z(y,t) &= \Big(\mathcal{G}^{i0}_{t,+}(z,y),\mathcal{G}^{i1}_{t,+}(z,y)\Big) \\
&= (\mathcal{G}^{i0}_t(z-y)-\mathcal{G}^{i0}_t(z-\tilde y),\mathcal{G}^{i1}_t(z-y)-\mathcal{G}^{i1}_t(z-\tilde y)),\quad i=0,1.
\end{aligned}
$$

The Parseval identity, formula (6.1), and condition **E6** imply that

$$
\|\Phi^i_z(\cdot,t)\|^2_{l^2} = (2\pi)^{-d}\int\limits_{\mathbb{T}^d}|\hat\Phi^i_z(\theta,t)|^2\,d\theta \le C\int\limits_{\mathbb{T}^d}\Big(|\hat{\mathcal{G}}^{i0}_t(\theta)|^2+|\hat{\mathcal{G}}^{i1}_t(\theta)|^2\Big)\,d\theta \le C_0<\infty,
$$

where the constant C_0 does not depend on $z\in\mathbb{Z}^d$ and $t\in\mathbb{R}$. Corollary 6.3 gives now

$$
|Q^{ij}_{\varepsilon,t}(z,z')| = |\langle Q_\varepsilon(y,y'),\Phi^i_z(y,t)\otimes\Phi^j_{z'}(y',t)\rangle_+| \le C\|\Phi^i_z(\cdot,t)\|_{l^2_+}\,\|\Phi^j_{z'}(\cdot,t)\|_{l^2_+} \le C_1<\infty,
$$

where the constant C_1 does not depend on $z,z'\in\mathbb{Z}^d_+$, $t\in\mathbb{R}$, and $\varepsilon>0$. ■

Appendix A: Proof of Lemma 6.7

By (6.13) and (6.15), the function $S^-_{\varepsilon,\tau/\varepsilon^2}$ can be represented as

$$
S^-_{\varepsilon,\tau/\varepsilon^2} = \frac{1}{2}\sum_{x\in\mathbb{Z}^d}\mathcal{G}^g_{\tau/\varepsilon^2}(l+x)\sum_{y\in\mathbb{Z}^d}\mathbf{R}_0(\kappa_{\varepsilon,x},y-x)\,\mathrm{sign}([r_1/\varepsilon]-y_1)\mathcal{G}^g_{\tau/\varepsilon^2}(p+y)^T,
$$

where, by definition, $\kappa_{\varepsilon,x}=\varepsilon[r/\varepsilon]-\varepsilon x\in\mathbb{R}^d$. The Parseval equality yields

$$
\begin{aligned}
&\sum_{y\in\mathbb{Z}^d}\mathrm{sign}([r_1/\varepsilon]-y_1)\mathbf{R}_0(\kappa_{\varepsilon,x},y-x)\mathcal{G}^g_{\tau/\varepsilon^2}(p+y)^T \\
&= (2\pi)^{-d}\int\limits_{\mathbb{T}^d}F_{y\to\theta}\Big[\mathrm{sign}([r_1/\varepsilon]-y_1)\mathbf{R}_0(\kappa_{\varepsilon,x},y-x)\Big]\overline{F_{y\to\theta}\Big[\mathcal{G}^g_{\tau/\varepsilon^2}(p+y)^T\Big]}\,d\theta.
\end{aligned}
$$

Note that

$$
F_{y\to\theta}\Big[\mathrm{sign}([r_1/\varepsilon]-y_1)\mathbf{R}_0(\kappa_{\varepsilon,x},y-x)\Big] = (2\pi)^{-d}F_{y\to\theta}\Big[\mathrm{sign}([r_1/\varepsilon]-y_1)\Big]*F_{y\to\theta}\Big[\mathbf{R}_0(\kappa_{\varepsilon,x},y-x)\Big],
$$

where $F_{y\to\theta}[\mathbf{R}_0(\kappa_{\varepsilon,x},y-x)]=e^{ix\cdot\theta}\hat{\mathbf{R}}_0(\kappa_{\varepsilon,x},\theta)$,

$$F_{y\to\theta}\Big[\,\mathrm{sign}([r_1/\varepsilon]-y_1)\Big]=-i\,(2\pi)^{d-1}\delta(\bar\theta)\mathrm{PV}\left(\frac{1}{\mathrm{tg}(\theta_1/2)}\right)e^{i[r_1/\varepsilon]\theta_1},$$

$\theta=(\theta_1,\bar\theta)$, $\bar\theta=(\theta_2,\dots,\theta_d)$, and PV stands for the Cauchy principal part. Hence,

$$\begin{aligned}S^-_{\varepsilon,\tau/\varepsilon^2} &= -\frac{i}{2}(2\pi)^{-d-1}\sum_{x\in\mathbb{Z}^d}\mathcal{G}^g_{\tau/\varepsilon^2}(l+x)\int\limits_{\mathbb{T}^d}\Big(\mathrm{PV}\int\limits_{\mathbb{T}^1}\frac{e^{i[r_1/\varepsilon](\theta_1-z)+ix_1z}}{\mathrm{tg}((\theta_1-z)/2)}\hat{\mathbf{R}}_0(\kappa_{\varepsilon,x},z,\bar\theta)dz\times\\ &\qquad\times e^{i\bar x\cdot\bar\theta}\hat{\mathcal{G}}^g_{\tau/\varepsilon^2}(\theta)^*e^{ip\cdot\theta}\Big)\,d\theta\\ &= -\frac{i}{2}(2\pi)^{-2d-1}\sum_{x\in\mathbb{Z}^d}\int\limits_{\mathbb{T}^{2d}}\Big(e^{-i(l+x)\cdot\theta'}\hat{\mathcal{G}}^g_{\tau/\varepsilon^2}(\theta')\times\\ &\qquad\times\mathrm{PV}\int\limits_{\mathbb{T}^1}\frac{e^{i[r_1/\varepsilon](\theta_1-z)+ix_1z}}{\mathrm{tg}((\theta_1-z)/2)}\hat{\mathbf{R}}_0(\kappa_{\varepsilon,x},z,\bar\theta)\,dz\,e^{i\bar x\cdot\bar\theta}\hat{\mathcal{G}}^g_{\tau/\varepsilon^2}(\theta)^*e^{ip\cdot\theta}\Big)\,d\theta d\theta'.\end{aligned}$$

We change variables $\theta_1\to\phi=\theta_1-z$, then denote $z=\theta_1$, and change variables $\theta'\to\varphi=\theta-\theta'$. Therefore,

$$S^-_{\varepsilon,\tau/\varepsilon^2}=(2\pi)^{-d}\frac{1}{4\pi i}\int\limits_{\mathbb{T}^d}e^{-i(l-p)\cdot\theta}I_\varepsilon(\theta)P_\varepsilon(\theta)\,d\theta, \qquad \text{(A.1)}$$

where $I_\varepsilon(\theta)$ is defined by (6.18), and $P_\varepsilon(\theta)$ stands for the matrix-valued function

$$P_\varepsilon(\theta)=\mathrm{PV}\int\limits_{\mathbb{T}^1}\frac{e^{i([r_1/\varepsilon]+p_1)\phi}}{\mathrm{tg}(\phi/2)}\hat{\mathcal{G}}^g_{\tau/\varepsilon^2}(\theta_1+\phi,\bar\theta)^*\,d\phi,\quad \theta=(\theta_1,\bar\theta).$$

Applying the partition of unity (6.4), (6.5), and formula (6.22), we rewrite $P_\varepsilon(\theta)$ in the form

$$P_\varepsilon(\theta)=\sum_{\sigma=1}^s\sum_{\pm}\mathrm{PV}\int\limits_{\mathbb{T}^1}\frac{e^{i([r_1/\varepsilon]+p_1)\phi}e^{\pm i\omega_\sigma(\theta_1+\phi,\bar\theta)\tau/\varepsilon^2}}{\mathrm{tg}(\phi/2)}\,g(\theta_1+\phi,\bar\theta)c^\mp_\sigma(\theta_1+\phi,\bar\theta)^*\,d\phi,$$

where $c^\mp_\sigma(\theta)^*=\Pi_\sigma(\theta)(I\mp iC^*_\sigma(\theta))/2$. Let us fix $r_1\in\mathbb{R}$, $\tau\neq0$. Since $\partial_1\omega_\sigma(\theta)\neq0$ for $\theta\in\mathrm{supp}\,g$, we can choose $\varepsilon_g\equiv\varepsilon_g(r_1,\tau)>0$ such that for all $\varepsilon<\varepsilon_g$ and $\theta\in\mathrm{supp}\,g$, $\partial_1\omega_\sigma(\theta)\neq\pm\varepsilon r_1/\tau$.

Lemma A.1 *Let us fix $r_1\in\mathbb{R}$ and $\tau\neq0$. Then*
(i) $\sup\limits_{\theta\in\mathbb{T}^d}\sup\limits_{\varepsilon>0}|P_\varepsilon(\theta)|<\infty$.
(ii) Let $r_1\pm\partial_1\omega_\sigma(\theta)\tau/\varepsilon\neq0$ for all $\theta\in\mathrm{supp}\,g$ and $\varepsilon\in(0,\varepsilon_g)$. Then

$$P_\varepsilon(\theta)-2\pi i\sum_{\sigma=1}^s\sum_{\pm}e^{\pm i\omega_\sigma(\theta)\tau/\varepsilon^2}\,\mathrm{sign}(r_1\pm\partial_1\omega_\sigma(\theta)\tau/\varepsilon)\,g(\theta)c^\mp_\sigma(\theta)^*\to0\quad \textit{as}\ \ \varepsilon\to+0.$$

Lemma A.1 can be proved by using the technique of [12, Lemma 8.3] or of [2, Proposition A.4 (i), (ii)]. The proof of this lemma is based on the following well-known assertion:

$$\lim_{\lambda\to+\infty}\left(\mathrm{PV}\int_{-\pi}^{\pi}\frac{e^{i\lambda\omega(z)}\chi(z)}{z}\,dz-\pi i e^{i\lambda\omega(0)}\chi(0)\,\mathrm{sign}\,\omega'(0)\right)=0,$$

where $\chi\in C^1$, $\omega\in C^2$, and $\omega'(0)\neq 0$.

By Lemma A.1 we can rewrite the r.h.s. of (A.1) in the form

$$S^-_{\varepsilon,\tau/\varepsilon^2}=(2\pi)^{-d}\frac{1}{2}\sum_{\sigma,\pm}\int_{\mathbb{T}^d}e^{-i(l-p)\cdot\theta}I_\varepsilon(\theta)e^{\pm i\omega_\sigma(\theta)\tau/\varepsilon^2}\,\mathrm{sign}(r_1\pm\partial_1\omega_\sigma(\theta)\tau/\varepsilon)h^\mp_\sigma(\theta)^*\,d\theta,\quad (A.2)$$

where $h^\mp_\sigma(\theta)=g(\theta)c^\mp_\sigma(\theta)$. Substituting (6.24) in (A.2) we obtain

$$\begin{aligned}S^-_{\varepsilon,\tau/\varepsilon^2}\;=\;&(2\pi)^{-d}\frac{1}{2}\int_{\mathbb{T}^d}e^{-i(l-p)\cdot\theta}\Big(\sum_{\sigma,\pm}h^\mp_\sigma(\theta)e^{\pm i\omega_\sigma(\theta)\tau/\varepsilon^2}A^\mp_{\varepsilon,\sigma}(\tau,r;\theta)\Big)\times\\&\times\Big(\sum_{\sigma',\pm}e^{\pm i\omega_{\sigma'}(\theta)\tau/\varepsilon^2}\,\mathrm{sign}(r_1\pm\partial_1\omega_{\sigma'}(\theta)\tau/\varepsilon)h^\mp_{\sigma'}(\theta)^*\Big)\,d\theta+o(1),\quad\varepsilon\to0.\ (A.3)\end{aligned}$$

Finally, comparing the r.h.s. of (A.3) and (6.26) we see that the problem of evaluating the limit value of (A.3) is solved by the similar way as in the step (v) of the proof of Lemma 6.6. ■

Appendix B: Proof of Lemma 6.8

By (6.15) we write

$$S^0_{\varepsilon,\tau/\varepsilon^2}=\sum_{x,y\in\mathbb{Z}^d}\mathcal{G}^g_{\tau/\varepsilon^2}(l+x)R^0(\kappa_{\varepsilon,x},[r/\varepsilon]-x,[r/\varepsilon]-y)\mathcal{G}^g_{\tau/\varepsilon^2}(p+y)^T,$$

where $\kappa_{\varepsilon,x}=\varepsilon[r/\varepsilon]-\varepsilon x$. Change variables $y\to z=y-x$ and denote the series over x by $\Phi_\varepsilon(z)$,

$$\begin{aligned}\Phi_\varepsilon(z)\;&\equiv\;\Phi_\varepsilon(z,\tau,r,l,p)\\&=\sum_{x\in\mathbb{Z}^d}\mathcal{G}^g_{\tau/\varepsilon^2}(l+x)R^0(\kappa_{\varepsilon,x},[r/\varepsilon]-x,[r/\varepsilon]-x-z)\mathcal{G}^g_{\tau/\varepsilon^2}(p+x+z)^T.\end{aligned}\quad (B.1)$$

Therefore,

$$S^0_{\varepsilon,\tau/\varepsilon^2}=\sum_{z\in\mathbb{Z}^d}\Phi_\varepsilon(z).\quad (B.2)$$

The estimate (4.9) for R and the notation (6.14) imply the same estimate for R^0,

$$|R^0(r,x,y)|\le C(1+|x-y|)^{-\gamma},\quad x,y\in\mathbb{Z}^d.\quad (B.3)$$

Next, the Cauchy–Schwartz inequality yields

$$\sum_{x\in\mathbb{Z}^d}|\mathcal{G}^g_{\tau/\varepsilon^2}(l+x)||\mathcal{G}^g_{\tau/\varepsilon^2}(p+x+z)^T|\le\|\mathcal{G}^g_{\tau/\varepsilon^2}(\cdot)\|^2_{\ell^2}\le C(1+\|\hat{V}^{-1}\|^2_{L^2(\mathbb{T}^d)}).\tag{B.4}$$

Hence, condition **E6** and the estimate (B.3) imply that $|\Phi_\varepsilon(z)|\le C(1+|z|)^{-\gamma}$. Since $\gamma>d$,

$$\sum_{z\in\mathbb{Z}^d}|\Phi_\varepsilon(z)|\le C<\infty,$$

and the series in (B.2) converges uniformly in ε (and also in τ,r,l,p). Therefore, it suffices to prove that

$$\lim_{\varepsilon\to 0}\Phi_\varepsilon(z)=0\quad\text{for each } z\in\mathbb{Z}^d.\tag{B.5}$$

Let us consider the series in (B.1). At first, note that by the property **(a)** for R and the definition (6.14), the function $R^0(r,x,y)$ has the form $R^0(r,x,y)=\mathbf{R}^0(r,x_1,y_1,\bar{x}-\bar{y})$, and $\mathbf{R}^0(r,x_1,y_1,\bar{z})=0$ for $y_1<0$. Hence,

$$\mathbf{R}^0(r,[r_1/\varepsilon]-x_1,[r_1/\varepsilon]-x_1-z_1,\bar{z})=0\quad\text{for } x_1>[r_1/\varepsilon]-z_1.$$

Secondly, it follows from condition (4.8) that $\forall\delta>0\ \ \exists K_\delta>0$ such that for any $y_1>K_\delta$ and $\forall r\in\mathbb{R}^d$, $|\mathbf{R}^0(r,y_1,y_1-z_1,\bar{z})|<\delta$. Hence, $\forall\delta>0\ \ \exists M_\delta=\max(K_\delta,z_1)>0$ such that

$$\begin{aligned}&\Big|\sum_{x\in\mathbb{Z}^d:\,x_1<[r_1/\varepsilon]-M_\delta}\mathcal{G}^g_{\tau/\varepsilon^2}(l+x)\mathbf{R}^0(\kappa_{\varepsilon,x},[r_1/\varepsilon]-x_1,[r_1/\varepsilon]-x_1-z_1,\bar{z})\mathcal{G}^g_{\tau/\varepsilon^2}(p+x+z)^T\Big|\\ \le\ &\delta\sum_{x\in\mathbb{Z}^d}\Big|\mathcal{G}^g_{\tau/\varepsilon^2}(l+x)\mathcal{G}^g_{\tau/\varepsilon^2}(p+x+z)^T\Big|\le C\delta,\end{aligned}$$

by the bound (B.4). Let us fix $\delta>0$. Therefore, it suffices to prove that for each $l,p,z\in\mathbb{Z}^d$,

$$\sum_{x_1\in A_\varepsilon}\sum_{\bar{x}\in\mathbb{Z}^{d-1}}\mathcal{G}^g_{\tau/\varepsilon^2}(l+x)\mathbf{R}^0(\kappa_{\varepsilon,x},[r_1/\varepsilon]-x_1,[r_1/\varepsilon]-x_1-z_1,\bar{z})\mathcal{G}^g_{\tau/\varepsilon^2}(p+x+z)^T\to 0\tag{B.6}$$

as $\varepsilon\to 0$, where $A_\varepsilon:=\{x_1\in\mathbb{Z}^1:\,x_1\in[[r_1/\varepsilon]-M_\delta,[r_1/\varepsilon]-z_1]\}$.

At first, note that if $x_1\in A_\varepsilon$, $[r_1/\varepsilon]-x_1\in[z_1,M_\delta]$. Denote $\bar{\kappa}_{\varepsilon,\bar{x}}\equiv\varepsilon[\bar{r}/\varepsilon]-\varepsilon\bar{x}$, $\bar{r}\in\mathbb{R}^{d-1}$, $\bar{x}\in\mathbb{Z}^{d-1}$. Then, by property (2.11) for $\mathbf{R}_0$ and property (4.10) for R,

$$\sup_{\bar{x}\in\mathbb{Z}^{d-1}}|\mathbf{R}^0(\varepsilon[r_1/\varepsilon]-\varepsilon x_1,\bar{\kappa}_{\varepsilon,\bar{x}},y_1,y_1',\bar{z})-\mathbf{R}^0(0,\bar{\kappa}_{\varepsilon,\bar{x}},y_1,y_1',\bar{z})|\to 0,\quad\varepsilon\to 0,$$

for every $x_1\in A_\varepsilon$, $y_1,y_1'\in\mathbb{Z}^1$, and $\bar{z}\in\mathbb{Z}^{d-1}$. Hence, by the bound (B.4), we can replace $\kappa_{\varepsilon,x}$ into $(0,\bar{\kappa}_{\varepsilon,\bar{x}})$ in the series (B.6).

Further, we change the variable $x_1\to x_1'$: $x_1'=[r_1/\varepsilon]-x_1$ in (B.6) and denote $x_1'=x_1$, $x_\varepsilon=([r_1/\varepsilon]-x_1,\bar{x})$. Therefore, to derive (B.6) it suffices to check that for any fixed $l,p,z\in\mathbb{Z}^d$, and $x_1\in[z_1,M_\delta]$,

$$D_\varepsilon:=\sum_{\bar{x}\in\mathbb{Z}^{d-1}}\mathcal{G}^g_{\tau/\varepsilon^2}(l+x_\varepsilon)\mathbf{R}^0(0,\bar{\kappa}_{\varepsilon,\bar{x}},x_1,x_1-z_1,\bar{z})\mathcal{G}^g_{\tau/\varepsilon^2}(p+x_\varepsilon+z)^T\to 0\quad\text{as }\varepsilon\to 0.$$

Let $d=1$. Then $D_\varepsilon=\mathcal{G}^g_{\tau/\varepsilon^2}(l_1+[r_1/\varepsilon]-x_1)\mathbf{R}^0(0,x_1,x_1-z_1)\mathcal{G}^g_{\tau/\varepsilon^2}(p_1+[r_1/\varepsilon]-x_1+z_1)^T$ vanishes as $\varepsilon\to 0$. Indeed, applying the inverse Fourier transform and the decomposition (6.22), we have

$$\begin{aligned}\mathcal{G}^g_{\tau/\varepsilon^2}(l_1+[r_1/\varepsilon]-x_1) &= \frac{1}{2\pi}\int_{\mathbb{T}^1} e^{-i(l_1+[r_1/\varepsilon]-x_1)\theta_1}\hat{\mathcal{G}}^g_{\tau/\varepsilon^2}(\theta_1)\,d\theta_1\\ &= \frac{1}{2\pi}\sum_{\sigma=1}^{s}\sum_{\pm}\int_{\mathbb{T}^1} e^{-i(l_1+[r_1/\varepsilon]-x_1)\theta_1}e^{\pm i\omega_\sigma(\theta_1)\tau/\varepsilon^2}g(\theta_1)c^{\mp}_\sigma(\theta_1)\,d\theta_1.\end{aligned}$$

The last integral vanishes as $\varepsilon\to 0$, since $\omega'_\sigma(\theta_1)\neq 0$ for all $\theta_1\in\operatorname{supp} g$.

Let $d>1$. In this case, we put, for simplicity, $\mathbf{R}^0(0,\bar\kappa_{\varepsilon,\bar x},\ldots)=\mathbf{R}^0(0,\bar\kappa_{\varepsilon,\bar x},x_1,x_1-z_1,\bar z)$. Using the Fourier transform we rewrite D_ε as

$$D_\varepsilon=\sum_{\bar x\in\mathbb{Z}^{d-1}}(2\pi)^{-2d}\int\limits_{\mathbb{T}^{2d}} e^{-i(l+x_\varepsilon)\cdot\theta'}\hat{\mathcal{G}}^g_{\tau/\varepsilon^2}(\theta')\mathbf{R}^0(0,\bar\kappa_{\varepsilon,\bar x},\ldots)e^{i(p+x_\varepsilon+z)\cdot\theta}\hat{\mathcal{G}}^g_{\tau/\varepsilon^2}(\theta)^*\,d\theta\,d\theta'.$$

Let us change variables $\theta=(\theta_1,\bar\theta)$, $\theta'=(\theta'_1,\bar\theta')$ as follows: $\bar\theta'\to\bar\varphi=\bar\theta-\bar\theta'$, and denote $\theta_1=\varphi_1$, $\theta'_1=\theta_1$. Hence,

$$D_\varepsilon=(2\pi)^{-d+1}\int\limits_{\mathbb{T}^{d-1}} e^{-i(\bar l-\bar p-\bar z)\cdot\bar\theta}A_\varepsilon(\bar\theta)B_\varepsilon(\bar\theta)\,d\bar\theta, \tag{B.7}$$

where

$$A_\varepsilon(\bar\theta) = \frac{1}{2\pi}\int_{\mathbb{T}^1} e^{-i(l_1+[r_1/\varepsilon]-x_1)\theta_1}\mathcal{I}_\varepsilon(\theta)\,d\theta_1, \tag{B.8}$$

$$B_\varepsilon(\bar\theta) = \frac{1}{2\pi}\int_{\mathbb{T}^1} e^{i(p_1+[r_1/\varepsilon]-x_1+z_1)\varphi_1}\hat{\mathcal{G}}^g_{\tau/\varepsilon^2}(\varphi_1,\bar\theta)^*\,d\varphi_1, \tag{B.9}$$

$$\mathcal{I}_\varepsilon(\theta) = (2\pi)^{-d+1}\int\limits_{\mathbb{T}^{d-1}} e^{i\bar l\cdot\bar\varphi}\hat{\mathcal{G}}^g_{\tau/\varepsilon^2}(\theta_1,\bar\theta-\bar\varphi)\Big(\sum_{\bar x\in\mathbb{Z}^{d-1}} e^{i\bar x\cdot\bar\varphi}\mathbf{R}^0(0,\varepsilon[\bar r/\varepsilon]-\varepsilon\bar x,\ldots)\Big)d\,\bar\varphi \tag{B.10}$$

(cf. formula (6.18)). Below we prove the following bound for $\mathcal{I}_\varepsilon(\theta)$ (cf. (6.24)):

$$\mathcal{I}_\varepsilon(\theta)=\sum_{\sigma=1}^{s}\sum_{\pm}h^{\mp}_\sigma(\theta)\,e^{\pm i\omega_\sigma(\theta)\tau/\varepsilon^2}\,\mathcal{A}^{\mp}_{\varepsilon,\sigma}(\tau,\bar r;\theta)+o(1),\quad \varepsilon\to 0, \tag{B.11}$$

where $h^{\mp}_\sigma(\theta)$ is defined in (6.22),

$$\mathcal{A}^{\mp}_{\varepsilon,\sigma}(\tau,\bar r;\theta)=\int\limits_{\mathbb{R}^{d-1}}\mathbf{R}^0(0,\bar r\mp\nabla_{\bar\theta}\,\omega_\sigma(\theta)\tau/\varepsilon-\bar y,\ldots)\mathcal{K}^{\mp}_\sigma(\tau,\bar y,\theta)\,d\bar y, \tag{B.12}$$

$\mathcal{K}^{\mp}_\sigma(\tau,\bar y,\theta)$, $\tau>0$, $\bar y\in\mathbb{R}^{d-1}$, $\theta\in\mathbb{T}^d\setminus\mathcal{C}$, stands for the following matrix-valued function

$$\mathcal{K}^{\mp}_\sigma(\tau,\bar y,\theta)=F^{-1}_{\bar\varphi\to\bar y}[e^{\pm i(\tau/2)\bar\varphi\cdot(\nabla^2_{\bar\theta}\omega_\sigma(\theta))\bar\varphi}]=\frac{e^{\pm i\pi s/4}}{(2\pi\,\tau)^{(d-1)/2}}\frac{e^{\mp i/(2\tau)\bar y\cdot(\bar\nabla^2\omega_\sigma(\theta))^{-1}\bar y}}{\sqrt{|\det(\nabla^2_{\bar\theta}\omega_\sigma(\theta))|}},$$

and s denotes the signature of the matrix $(\nabla^2_{\bar\theta}\omega_\sigma(\theta))$, $\theta=(\theta_1,\bar\theta)\in\mathbb{T}^d\setminus\mathcal{C}$ (cf. (6.25)).

To prove (B.11) we first apply the Poisson summation formula (see (6.19)) and obtain

$$\sum_{\bar x\in\mathbb{Z}^{d-1}} e^{i\bar x\cdot\bar\varphi}\mathbf{R}^0(0,\varepsilon[\bar r/\varepsilon]-\varepsilon\bar x,\ldots)=\varepsilon^{-d+1}\sum_{\bar n\in\mathbb{Z}^{d-1}}\tilde{\mathbf{R}}^0(0,-\bar\varphi/\varepsilon-2\pi\bar n/\varepsilon,\ldots)e^{i[\bar r/\varepsilon]\cdot\bar\varphi},$$

where $\tilde{\mathbf{R}}^0(0,\bar\varphi,\ldots)$ stands for the Fourier transform $\tilde{\mathbf{R}}^0(0,\bar\varphi,\ldots)=F_{\bar r\to\bar\varphi}[\mathbf{R}^0(0,\bar r,\ldots)]$, $\bar r,\bar\varphi\in\mathbb{R}^{d-1}$. Conditions (2.11) and (4.11) yield

$$\varepsilon^{-d+1}\sum_{\bar n\neq 0}\tilde{\mathbf{R}}^0(0,-\bar\varphi/\varepsilon-2\pi\bar n/\varepsilon,\ldots)\to 0\quad\text{as }\ \varepsilon\to 0 \tag{B.13}$$

uniformly in $\bar\varphi\in[-\pi,\pi]^{d-1}$. Hence, by (B.10) and (B.13),

$$\begin{aligned}\mathcal{I}_\varepsilon(\theta)&=(2\pi\varepsilon)^{-d+1}\int\limits_{[-\pi,\pi]^{d-1}} e^{i\bar\varphi\cdot(\bar l+[\bar r/\varepsilon])}\hat{\mathcal{G}}^g_{\tau/\varepsilon^2}(\theta_1,\bar\theta-\bar\varphi)\tilde{\mathbf{R}}^0(0,-\bar\varphi/\varepsilon,\ldots)\,d\bar\varphi+o(1)\\ &=(2\pi)^{-d+1}\int\limits_{[-\pi/\varepsilon,\pi/\varepsilon]^{d-1}} e^{-i\varepsilon\bar\varphi\cdot(\bar l+[\bar r/\varepsilon])}\hat{\mathcal{G}}^g_{\tau/\varepsilon^2}(\theta_1,\bar\theta+\varepsilon\bar\varphi)\tilde{\mathbf{R}}^0(0,\bar\varphi,\ldots)\,d\bar\varphi+o(1).\end{aligned}$$

Secondly, using the representation (6.22) for $\hat{\mathcal{G}}^g_t(\theta)$ we rewrite $\mathcal{I}_\varepsilon(\theta)$ in the form

$$\begin{aligned}\mathcal{I}_\varepsilon(\theta)&=(2\pi)^{-d+1}\sum_{\sigma,\pm}\int\limits_{[-\pi/\varepsilon,\pi/\varepsilon]^{d-1}} e^{-i\varepsilon\bar\varphi\cdot(\bar l+[\bar r/\varepsilon])}e^{\pm i\omega_\sigma(\theta_1,\bar\theta+\varepsilon\bar\varphi)\tau/\varepsilon^2}h^\mp_\sigma(\theta_1,\bar\theta+\varepsilon\bar\varphi)\tilde{\mathbf{R}}^0(0,\bar\varphi,\ldots)\,d\bar\varphi\\ &\quad+o(1)=(2\pi)^{-d+1}\sum_{\sigma,\pm}h^\mp_\sigma(\theta)\int\limits_{\mathbb{R}^{d-1}} e^{-i\bar\varphi\cdot\bar r}e^{\pm i\omega_\sigma(\theta_1,\bar\theta+\varepsilon\bar\varphi)\tau/\varepsilon^2}\tilde{\mathbf{R}}^0(0,\bar\varphi,\ldots)\,d\bar\varphi+o(1),\end{aligned}$$

by conditions (2.11) and (4.11). Finally, we use the Taylor sum representation for $\omega_\sigma(\theta_1,\bar\theta+\varepsilon\bar\varphi)$ (cf (6.23)) and obtain the bound (B.11), since

$$\begin{aligned}&(2\pi)^{-d+1}\int\limits_{\mathbb{R}^{d-1}} e^{-i\bar\varphi\cdot(\bar r\mp\nabla_{\bar\theta}\omega_\sigma(\theta)\tau/\varepsilon)}e^{\pm i(\tau/2)\bar\varphi\cdot(\nabla^2_{\bar\theta}\omega_\sigma(\theta))\bar\varphi}\tilde{\mathbf{R}}^0(0,\bar\varphi,\ldots)\,d\bar\varphi\\ &=F^{-1}_{\bar\varphi\to\bar r\mp\nabla_{\bar\theta}\omega_\sigma(\theta)\tau/\varepsilon}[\tilde{\mathbf{R}}^0(0,\bar\varphi,\ldots)e^{\pm i(\tau/2)\bar\varphi\cdot(\nabla^2_{\bar\theta}\omega_\sigma(\theta))\bar\varphi}]=\mathcal{A}^\mp_{\varepsilon,\sigma}(\tau,\bar r;\theta),\end{aligned}$$

where $\mathcal{A}^\mp_{\varepsilon,\sigma}(\tau,\bar r;\theta)$ is defined in (B.12). The bound (B.11) is proved.

Now we prove that the r.h.s. of (B.7) vanishes as $\varepsilon\to 0$, using the Lebesgue dominated convergence theorem. At first, we show that

$$|A_\varepsilon(\bar\theta)B_\varepsilon(\bar\theta)|\le v(\bar\theta),\quad\forall\varepsilon>0,\quad\text{where }\ v(\bar\theta)\in L^1(\mathbb{T}^{d-1}).$$

Indeed, by conditions **I4'** and (4.11),

$$\sup_{\varepsilon>0}\sup_{\theta\in\mathbb{T}^d}|\mathcal{A}^\mp_{\varepsilon,\sigma}(\tau,\bar r;\theta)|\le C<\infty.$$

Since $h^{\mp}_\sigma(\theta)=g(\theta)\Pi_\sigma(\theta)(I\mp iC_\sigma(\theta))/2$, the bound (B.11) yields

$$|\mathcal{I}_\varepsilon(\theta)|\le\sum_{\sigma=1}^{s}(C_1\omega_\sigma^{-1}(\theta)+C_2\omega_\sigma(\theta))+C_3,\quad \theta\in\mathbb{T}^d\setminus\mathcal{C}.$$

Write $v(\bar\theta):=\int\limits_{\mathbb{T}^1}(1+\|\hat V^{-1}(\theta)\|)\,d\theta_1$, $\bar\theta\in\mathbb{T}^{d-1}$. Therefore, by (B.8) and (B.9),

$$\begin{aligned}|A_\varepsilon(\bar\theta)| &\le C\int\limits_{\mathbb{T}^1}|\mathcal{I}_\varepsilon(\theta)|\,d\theta_1\le C_1(v(\bar\theta))^{1/2},\\ |B_\varepsilon(\bar\theta)| &\le C\int\limits_{\mathbb{T}^1}|\hat{\mathcal{G}}^g_{\tau/\varepsilon^2}(\theta)|\,d\theta_1\le C_1(v(\bar\theta))^{1/2},\end{aligned}$$

and, evidently, $|A_\varepsilon(\bar\theta)B_\varepsilon(\bar\theta)|\le C\,v(\bar\theta)$, where $v(\bar\theta)\in L^1(\mathbb{T}^{d-1})$ by condition **E6**. Therefore, to prove that the integral in (B.7) tends to zero it suffices to check that $B_\varepsilon(\bar\theta)$ vanishes as $\varepsilon\to 0$ for a.a. $\bar\theta\in\mathbb{T}^{d-1}$. By the representation (6.22),

$$B_\varepsilon(\bar\theta)=\frac{1}{2\pi}\sum_{\sigma=1}^{s}\sum_{\pm}\int\limits_{\mathbb{T}^1}e^{i(p_1+[r_1/\varepsilon]-x_1+z_1)\varphi_1}e^{\pm i\omega_\sigma(\varphi_1,\bar\theta)\tau/\varepsilon^2}h^{\mp}_\sigma(\varphi_1,\bar\theta)^*\,d\varphi_1.$$

For fixed $r_1\in\mathbb{R}^1$, $\tau\ne 0$, and $p_1,x_1,z_1\in\mathbb{Z}^1$, $\operatorname{mes}\{\varphi_1\in\mathbb{T}^1:\partial_1\omega_\sigma(\varphi_1,\bar\theta)=0\}=0$ for a.a. fixed $\bar\theta\in\mathbb{T}^{d-1}$. Hence, $B_\varepsilon(\bar\theta)$ vanishes as $\varepsilon\to0$. It can be proved similarly to the Lebesgue–Riemann theorem. ■

Appendix C: Proof of Lemma 6.6 in the general case

We prove Lemma 6.6 under the weaker condition (2.13)–(2.14) on $\hat{\mathbf{R}}_0(r,\theta)$ than (2.11).

Step (i) of the proof is not changed and we derive formulas (6.16)–(6.18).

Next step is to apply the Poisson summation formula (6.19). However, in general, we can not apply (6.19) to $\hat{\mathbf{R}}_0(r,\theta)$. Therefore, introduce

$$\hat{\mathbf{R}}_{r,\varepsilon}(y,\theta)=\hat{\mathbf{R}}_0(y,\theta)e^{-\varepsilon^\beta(y-r)^2},\quad y\in\mathbb{R}^d,\quad \theta\in\mathbb{T}^d, \tag{C.1}$$

with $\beta\in(d+2,2(\delta-1-d)/d)$ and δ from (2.14). The asymptotics of (6.18) is not changed if we replace the function $\hat{\mathbf{R}}_0(y,\theta)$ by $\hat{\mathbf{R}}_{r,\varepsilon}(y,\theta)$ in (6.18). Indeed, consider the following series:

$$\begin{aligned}&\sum_{x\in\mathbb{Z}^d}\int\limits_{\mathbb{T}^d}e^{i(l+x)\cdot\varphi}\hat{\mathcal{G}}^g_{\tau/\varepsilon^2}(\theta-\varphi)\,d\varphi\Big(\hat{\mathbf{R}}_0(\varepsilon[r/\varepsilon]-\varepsilon x,\theta)-\hat{\mathbf{R}}_{r,\varepsilon}(\varepsilon[r/\varepsilon]-\varepsilon x,\theta)\Big)\\ =&\sum_{|x|\le c\tau/\varepsilon^2}\ldots+\sum_{|x|\ge c\tau/\varepsilon^2}\ldots=A_\varepsilon(\theta)+B_\varepsilon(\theta).\end{aligned}$$

and prove that the series $A_\varepsilon(\theta)$ and $B_\varepsilon(\theta)$ vanish as $\varepsilon\to 0$. For the first series $A_\varepsilon(\theta)$, we apply Lemma 6.4 (i), definition (C.1) and property **I1** and obtain

$$\begin{aligned}
A_\varepsilon(\theta) &\le C\sum_{|x|\le c\tau/\varepsilon^2}\Big|\int_{\mathbb{T}^d} e^{i(l+x)\cdot\varphi}\hat{\mathcal{G}}^g_{\tau/\varepsilon^2}(\theta-\varphi)\,d\varphi\Big|\Big|\hat{\mathbf{R}}_0(\varepsilon[r/\varepsilon]-\varepsilon x,\theta)-\hat{\mathbf{R}}_{r,\varepsilon}(\varepsilon[r/\varepsilon]-\varepsilon x,\theta)\Big|\\
&\le C_1\sum_{|x|\le c\tau/\varepsilon^2}|\mathcal{G}^g_{\tau/\varepsilon^2}(l+x)|\varepsilon^\beta(\varepsilon|x|+\varepsilon)^2\le C\varepsilon^d\varepsilon^{\beta+2}(\tau/\varepsilon^2)^{2+d}\to 0,\quad \varepsilon\to 0,
\end{aligned}$$

uniformly in θ, since $\beta>2+d$. For the second series $B_\varepsilon(\theta)$, Lemma 6.4 (ii) yields

$$\begin{aligned}
B_\varepsilon(\theta) &\le C\sum_{|x|\ge c\tau/\varepsilon^2}\Big|\int_{\mathbb{T}^d} e^{i(l+x)\cdot\varphi}\hat{\mathcal{G}}^g_{\tau/\varepsilon^2}(\theta-\varphi)\,d\varphi\Big||\hat{\mathbf{R}}_0(\varepsilon[r/\varepsilon]-\varepsilon x,\theta)|\\
&\le C_1\sum_{|x|\ge c\tau/\varepsilon^2}\Big|\mathcal{G}^g_{\tau/\varepsilon^2}(l+x)\Big|\to 0,\quad \varepsilon\to 0.
\end{aligned}$$

Step (ii): Now we apply formula (6.19) to $\sum\limits_{x\in\mathbb{Z}^d} e^{ix\cdot\varphi}\hat{\mathbf{R}}_{r,\varepsilon}(\varepsilon[r/\varepsilon]-\varepsilon x,\theta)$. By $\tilde{\mathbf{R}}_{r,\varepsilon}(s,\theta)$, $s\in\mathbb{R}^d$, $\theta\in\mathbb{T}^d$, we denote the Fourier transform of $\hat{\mathbf{R}}_{r,\varepsilon}(y,\theta)$ w.r.t. y (see (2.12)) and obtain

$$I_\varepsilon(\theta)=(2\pi\varepsilon)^{-d}\int_{\mathbb{T}^d} e^{i(l+[r/\varepsilon])\cdot\varphi}\hat{\mathcal{G}}^g_{\tau/\varepsilon^2}(\theta-\varphi)\Big(\sum_{n\in\mathbb{Z}^d}\tilde{\mathbf{R}}_{r,\varepsilon}(-\varphi/\varepsilon-2\pi n/\varepsilon,\theta)\Big)\,d\varphi+o(1)\quad\text{(C.2)}$$

(cf (6.20)). We check that the contribution of the series with $n\ne 0$ in (C.2) vanishes,

$$\varepsilon^{-d}\sum_{n\in\mathbb{Z}^d,n\ne 0}\tilde{\mathbf{R}}_{r,\varepsilon}(-\varphi/\varepsilon-2\pi n/\varepsilon,\theta)\to 0\quad\text{as }\varepsilon\to 0,\qquad\text{(C.3)}$$

uniformly in $\theta\in\mathbb{T}^d$, $\varphi\in[-\pi,\pi]^d$ (cf (6.21)). At first, we derive the formula for $\tilde{\mathbf{R}}_{r,\varepsilon}(\xi,\theta)$:

$$\tilde{\mathbf{R}}_{r,\varepsilon}(\xi,\theta)=e^{ir\cdot\xi}(4\pi k)^{-d/2}\int_{\mathbb{R}^d} e^{-is\cdot r}e^{-|s-\xi|^2/(4k)}\mu(\theta,ds),\quad \xi\in\mathbb{R}^d,\quad\theta\in\mathbb{T}^d,\qquad\text{(C.4)}$$

where $k:=\varepsilon^\beta$ and $\mu(\theta,ds)$ from (2.13). Indeed, by definition (C.1),

$$\tilde{\mathbf{R}}_{r,\varepsilon}(\xi,\theta)=\int_{\mathbb{R}^d} e^{i\xi\cdot y-k(y-r)^2}\hat{\mathbf{R}}_0(y,\theta)\,dy=e^{ir\cdot\xi-|\xi|^2/(4k)}\int_{\mathbb{R}^d} e^{-k(y-r-i\xi/(2k))^2}\hat{\mathbf{R}}_0(y,\theta)\,dy.$$

On the other hand, by definition (2.13), for any $\alpha\in\mathbb{R}^d$,

$$\begin{aligned}
\int_{\mathbb{R}^d} e^{-k(y-\alpha)^2}\hat{\mathbf{R}}_0(y,\theta)\,dy &= (2\pi)^{-d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} e^{-iy\cdot s}e^{-k(y-\alpha)^2}\,\mu(\theta,ds)\,dy\\
&= (2\pi)^{-d}\int_{\mathbb{R}^d} e^{-is\cdot\alpha}\left(\frac{\pi}{k}\right)^{d/2}e^{-|s|^2/(4k)}\,\mu(\theta,ds).
\end{aligned}$$

Hence,

$$\tilde{\mathbf{R}}_{r,\varepsilon}(\xi,\theta) = e^{ir\cdot\xi - |\xi|^2/(4k)}(4\pi k)^{-d/2}\int\limits_{\mathbb{R}^d} e^{-is\cdot(r+i\xi/(2k))}e^{-|s|^2/(4k)}\,\mu(\theta,ds).$$

This implies formula (C.4).

Now we prove (C.3). Let us set $\xi_{n,\varepsilon} = -\varphi/\varepsilon - 2\pi n/\varepsilon$, where $|\varphi_i| \le \pi$, $i = 1,\dots,d$, apply (C.4) and devide the integration into two: $|s| \le \pi/(2\varepsilon)$ and $|s| \ge \pi/(2\varepsilon)$:

$$\begin{aligned}\varepsilon^{-d}\Big|\sum_{n\ne 0}\tilde{\mathbf{R}}_{r,\varepsilon}(\xi_{n,\varepsilon},\theta)\Big| &\le \varepsilon^{-d}\sum_{n\ne 0}(4\pi k)^{-d/2}\Big|\int\limits_{\mathbb{R}^d} e^{-is\cdot r}e^{-|s-\xi_{n,\varepsilon}|^2/(4k)}\mu(\theta,ds)\Big| \\ &= I_1 + I_2, \end{aligned} \tag{C.5}$$

where

$$\begin{aligned} I_1 &= (\varepsilon\sqrt{4\pi k})^{-d}\sum_{n\ne 0}\Big|\int\limits_{|s|\le\pi/(2\varepsilon)} e^{-is\cdot r}e^{-|s-\xi_{n,\varepsilon}|^2/(4k)}\mu(\theta,ds)\Big|, \\ I_2 &= (\varepsilon\sqrt{4\pi k})^{-d}\sum_{n\ne 0}\Big|\int\limits_{|s|\ge\pi/(2\varepsilon)} e^{-is\cdot r}e^{-|s-\xi_{n,\varepsilon}|^2/(4k)}\mu(\theta,ds)\Big|. \end{aligned} \tag{C.6}$$

Let $|s| \le \pi/(2\varepsilon)$. Put $\xi_{n_i,\varepsilon} = -\varphi_i/\varepsilon - 2\pi n_i/\varepsilon$. Then $|s_i - \xi_{n_i,\varepsilon}| \ge |2\pi|n_i|/\varepsilon - |s_i + \varphi_i/\varepsilon|| \ge (\pi/\varepsilon)(2|n_i| - 3/2)$ if $n_i \ne 0$. Therefore, there exist positive constants a, b such that

$$\sum_{n_i\in\mathbb{Z}^1:n_i\ne 0} e^{-(s_i-\xi_{n_i,\varepsilon})^2/(4k)} \le \sum_{n_i\ne 0} e^{-\pi^2(2|n_i|-3/2)^2/(4k\varepsilon^2)} \le a\,e^{-b/(k\varepsilon^2)}\,\sqrt{k}\varepsilon, \quad i = 1,\dots,d.$$

Hence, there exist positive constants a_s and b_s such that

$$I_1 \sim \sum_{s=0}^{d-1} a_s\,e^{-b_s/(k\varepsilon^2)}(\sqrt{k}\varepsilon)^{-s} \to 0, \quad \varepsilon\to 0,$$

since $k = \varepsilon^\beta$ with $\beta > 0$. Let $|s| \ge \pi/(2\varepsilon)$. Then

$$\sum_{n\in\mathbb{Z}^d} e^{-(s-\xi_{n,\varepsilon})^2/(4k)} = \sum_{n\in\mathbb{Z}^d} e^{-(\pi^2/(k\varepsilon^2))(n+\varphi/(2\pi)+s\varepsilon/(2\pi))^2} \le C_1(\varepsilon\sqrt{k})^d + C_2. \tag{C.7}$$

Using (C.6), (C.7), and condition (2.14), we obtain

$$I_2 \sim (C_1 + C_2(\varepsilon\sqrt{k})^{-d})\sup_{\theta\in\mathbb{T}^d}\int\limits_{|s|\ge\pi/(2\varepsilon)} |\mu(\theta,ds)| \le C_3\varepsilon^{\delta-1-(\beta/2+1)d} \to 0, \quad \varepsilon\to 0,$$

because $k = \varepsilon^\beta$ and $\delta > 1 + (\beta/2+1)d$. This completes the proof of the bound (C.3).

Step (iii): We apply (C.2), (C.3), change variables $\varphi \to -\varepsilon\varphi$ and use formula (6.22):

$$I_\varepsilon(\theta) \;=\; (2\pi\varepsilon)^{-d}\int\limits_{[-\pi,\pi]^d} e^{i\varphi\cdot(l+[r/\varepsilon])}\hat{\mathcal{G}}^g_{\tau/\varepsilon^2}(\theta-\varphi)\tilde{\mathbf{R}}_{r,\varepsilon}(-\varphi/\varepsilon,\theta)\,d\varphi + o(1)$$

$$= (2\pi)^{-d} \int\limits_{[-\pi/\varepsilon,\pi/\varepsilon]^d} e^{-i\varepsilon\varphi\cdot(l+[r/\varepsilon])} \hat{\mathcal{G}}^g_{\tau/\varepsilon^2}(\theta+\varepsilon\varphi)\tilde{\mathbf{R}}_{r,\varepsilon}(\varphi,\theta)\,d\varphi + o(1)$$

$$= (2\pi)^{-d}\sum_{\sigma,\pm} \int\limits_{[-\pi/\varepsilon,\pi/\varepsilon]^d} e^{-i\varepsilon\varphi\cdot(l+[r/\varepsilon])} e^{\pm i\omega_\sigma(\theta+\varepsilon\varphi)\tau/\varepsilon^2} h^{\mp}_\sigma(\theta+\varepsilon\varphi)\tilde{\mathbf{R}}_{r,\varepsilon}(\varphi,\theta)\,d\varphi + o(1). \quad \text{(C.8)}$$

The asymptotics of (C.8) is not changed if we replace $e^{-i\varepsilon\varphi\cdot(l+[r/\varepsilon])}$ by $e^{-i\varphi\cdot r}$ in the integrand. Indeed, we check that the integral

$$D_\varepsilon(\theta) = \int\limits_{[-\pi/\varepsilon,\pi/\varepsilon]^d} (e^{-i\varepsilon\varphi\cdot(l+[r/\varepsilon])} - e^{-i\varphi\cdot r}) e^{\pm i\omega_\sigma(\theta+\varepsilon\varphi)\tau/\varepsilon^2} h^{\mp}_\sigma(\theta+\varepsilon\varphi)\tilde{\mathbf{R}}_{r,\varepsilon}(\varphi,\theta)\,d\varphi$$

vanishes as $\varepsilon \to 0$. We apply formula (C.4) and change the order of the integration $\mu(\theta,ds)\,d\varphi \to d\varphi\,\mu(\theta,ds)$:

$$\begin{aligned} D_\varepsilon(\theta) &= \int\limits_{[-\pi/\varepsilon,\pi/\varepsilon]^d} (e^{-i\varepsilon\varphi\cdot(l+[r/\varepsilon])} - e^{-i\varphi\cdot r}) e^{\pm i\omega_\sigma(\theta+\varepsilon\varphi)\tau/\varepsilon^2} h^{\mp}_\sigma(\theta+\varepsilon\varphi) \times \\ &\quad \times\Big(\frac{e^{ir\cdot\varphi}}{(4\pi k)^{d/2}} \int\limits_{\mathbb{R}^d} e^{-is\cdot r} e^{-|s-\varphi|^2/(4k)} \mu(\theta,ds)\Big)\,d\varphi = \int\limits_{\mathbb{R}^d} e^{-is\cdot r} C_\varepsilon(s,\theta)\,\mu(\theta,ds), \end{aligned} \quad \text{(C.9)}$$

where $C_\varepsilon(s,\theta)$ stands for the inner integral in $d\varphi$:

$$C_\varepsilon(s,\theta) = \frac{1}{(4\pi k)^{d/2}} \int\limits_{[-\pi/\varepsilon,\pi/\varepsilon]^d} e^{-|s-\varphi|^2/(4k)} (e^{i\varphi\cdot(r-\varepsilon[r/\varepsilon]-\varepsilon l)} - 1) e^{\pm i\omega_\sigma(\theta+\varepsilon\varphi)\tau/\varepsilon^2} h^{\mp}_\sigma(\theta+\varepsilon\varphi)\,d\varphi.$$

Note that

$$|C_\varepsilon(s,\theta)| \le \varepsilon \sup_{\theta\in\mathbb{T}^d} |h^{\mp}_\sigma(\theta)| \frac{1}{(4\pi k)^{d/2}} \int\limits_{[-\pi/\varepsilon,\pi/\varepsilon]^d} e^{-|s-\varphi|^2/(4k)} |\varphi|\,d\varphi,$$

where

$$\frac{1}{(4\pi k)^{d/2}} \int\limits_{[-\pi/\varepsilon,\pi/\varepsilon]^d} e^{-|s-\varphi|^2/(4k)}|\varphi|\,d\varphi \le \frac{1}{(4\pi k)^{d/2}} \int\limits_{\mathbb{R}^d} e^{-|\varphi|^2/(4k)}(|s|+|\varphi|)\,d\varphi \le C_1|s| + C_2\sqrt{k}.$$

Hence, by (C.9) and (2.14), $\sup_{\theta\in\mathbb{T}^d}|D_\varepsilon(\theta)| \le C\varepsilon$. Similarly, the asymptotics of (C.8) is not changed if we replace $h^{\mp}_\sigma(\theta+\varepsilon\varphi)$ by $h^{\mp}_\sigma(\theta)$ in the integrand. Therefore,

$$I_\varepsilon(\theta) = \sum_{\sigma,\pm} h^{\mp}_\sigma(\theta) N^{\mp}_{\varepsilon,\sigma}(\tau,r;\theta) + o(1), \quad \varepsilon \to 0, \quad \text{(C.10)}$$

where $N^{\mp}_{\varepsilon,\sigma}(\tau,r;\theta)$ stands for the matrix-valued function,

$$N^{\mp}_{\varepsilon,\sigma}(\tau,r;\theta) = (2\pi)^{-d} \int\limits_{[-\pi/\varepsilon,\pi/\varepsilon]^d} e^{-i\varphi\cdot r} e^{\pm i\omega_\sigma(\theta+\varepsilon\varphi)\tau/\varepsilon^2} \tilde{\mathbf{R}}_{r,\varepsilon}(\varphi,\theta)\,d\varphi. \quad \text{(C.11)}$$

Step (iv): Now we apply the Taylor sum representation (cf (6.23)) and replace $\omega_\sigma(\theta+\varepsilon\varphi)$ by $\omega_\sigma(\theta)+\varepsilon\nabla\omega_\sigma(\theta)\cdot\varphi+(\varepsilon^2/2)\varphi\cdot H_\sigma(\theta)\varphi$. To do it we check that

$$L_\varepsilon(\theta):=\int\limits_{[-\pi/\varepsilon,\pi/\varepsilon]^d} e^{-i\varphi\cdot r}\Big[e^{\pm i\omega_\sigma(\theta+\varepsilon\varphi)\tau/\varepsilon^2}-e^{\pm i(\omega_\sigma(\theta)+\varepsilon\nabla\omega_\sigma(\theta)\cdot\varphi+\frac{\varepsilon^2}{2}\varphi\cdot H_\sigma(\theta)\varphi)\tau/\varepsilon^2}\Big]\tilde{\mathbf{R}}_{r,\varepsilon}(\varphi,\theta)\,d\varphi\to 0$$

as $\varepsilon\to 0$. As in *Step (iii)*, we use formula (C.4) and change the order of the integration $\mu(\theta,ds)\,d\varphi\to d\varphi\,\mu(\theta,ds)$. Therefore,

$$L_\varepsilon(\theta)=\int\limits_{\mathbb{R}^d} e^{-is\cdot r}D_\varepsilon(s,\theta)\,\mu(\theta,ds),$$

where $D_\varepsilon(s,\theta)$ stands for the inner integral in $d\varphi$:

$$D_\varepsilon(s,\theta)=\frac{1}{(4\pi k)^{d/2}}\int\limits_{[-\pi/\varepsilon,\pi/\varepsilon]^d} e^{-|s-\varphi|^2/(4k)}\Big[e^{\pm i\omega_\sigma(\theta+\varepsilon\varphi)\tau/\varepsilon^2}-e^{\pm i(\omega_\sigma(\theta)+\varepsilon\nabla\omega_\sigma(\theta)\cdot\varphi+\frac{\varepsilon^2}{2}\varphi\cdot H_\sigma(\theta)\varphi)\tau/\varepsilon^2}\Big]d\varphi.$$

Moreover, by condition (2.14),

$$\begin{aligned}\sup_{\theta\in\mathbb{T}^d}\Big|\int\limits_{|s|\ge N} e^{-is\cdot r}D_\varepsilon(s,\theta)\,\mu(\theta,ds)\Big| &\le \sup_{\theta\in\mathbb{T}^d}\int\limits_{|s|\ge N}\Big(\frac{2}{(4\pi k)^{d/2}}\int\limits_{\mathbb{R}^d} e^{-|s-\varphi|^2/(4k)}d\varphi\Big)|\mu|(\theta,ds)\\ &\le C(1+N)^{-\delta+1}\to 0\quad\text{as } N\to\infty,\end{aligned}$$

uniformly in ε. Hence, it suffices to prove that for any fixed $N\in\mathbb{N}$,

$$L^N_\varepsilon(\theta):=\int\limits_{|s|\le N} e^{-is\cdot r}D_\varepsilon(s,\theta)\,\mu(\theta,ds)\to 0\quad\text{as } \varepsilon\to 0. \tag{C.12}$$

To check (C.12) we estimate the inner integral $D_\varepsilon(s,\theta)$:

$$\begin{aligned}|D_\varepsilon(s,\theta)| &\le C\varepsilon\tau\frac{1}{(4\pi k)^{d/2}}\int\limits_{[-\pi/\varepsilon,\pi/\varepsilon]^d} e^{-|s-\varphi|^2/(4k)}|\varphi|^3\,d\varphi\\ &\le C_1\varepsilon\frac{1}{(4\pi k)^{d/2}}\Big[\int\limits_{\mathbb{R}^d} e^{-|\varphi|^2/(4k)}|\varphi|^3\,d\varphi+|s|^3\int\limits_{\mathbb{R}^d} e^{-|\varphi|^2/(4k)}\,d\varphi\Big]\le C_2\varepsilon[k^{3/2}+|s|^3].\end{aligned}$$

Hence, for fixed N,

$$|L^N_\varepsilon(\theta)|\le C\varepsilon\int\limits_{|s|\le N}(\varepsilon^{3\beta/2}+|s|^3)|\mu|(\theta,ds)\le C_2(N)\varepsilon,$$

and (C.12) follows. Therefore, $\sup_{\theta\in\mathbb{T}^d}|L_\varepsilon(\theta)|\to 0$ as $\varepsilon\to 0$, and

$$N^{\mp}_{\varepsilon,\sigma}(\tau,r;\theta)=e^{\pm i\omega_\sigma(\theta)\tau/\varepsilon^2}B^{\mp}_{\varepsilon,\sigma}(\tau,r;\theta)+o(1),\quad \varepsilon\to 0, \tag{C.13}$$

where, by definition,

$$B^{\mp}_{\varepsilon,\sigma}(\tau,r;\theta) \;=\; (2\pi)^{-d}\int\limits_{\mathbb{R}^d} e^{-i\varphi\cdot(r\mp\nabla\omega_\sigma(\theta)\tau/\varepsilon)}e^{\pm i(\tau/2)\varphi\cdot H_\sigma(\theta)\varphi}\tilde{\mathbf{R}}_{r,\varepsilon}(\varphi,\theta)\,d\varphi$$
$$= \int\limits_{\mathbb{R}^d}\hat{\mathbf{R}}_{r,\varepsilon}(r\mp\nabla\omega_\sigma(\theta)\tau/\varepsilon - x,\theta)K^{\mp}_\sigma(\tau,x,\theta)\,dx, \tag{C.14}$$

with the function $K^{\mp}_\sigma(\tau,x,\theta)$ ($\tau>0$, $x\in\mathbb{R}^d$, $\theta\in\mathbb{T}^d\setminus\mathcal{C}$) from (6.25).

Remark. The integral $N^{\mp}_{\varepsilon,\sigma}(\tau,r;\theta)$ can be rewritten in the another form. Namely,

$$N^{\mp}_{\varepsilon,\sigma}(\tau,r;\theta) = e^{\pm i\omega_\sigma(\theta)\tau/\varepsilon^2}C^{\mp}_{\varepsilon,\sigma}(\tau,r;\theta) + o(1),\quad \varepsilon\to 0, \tag{C.15}$$

where, by definition,

$$C^{\mp}_{\varepsilon,\sigma}(\tau,r;\theta) := (2\pi)^{-d}\int\limits_{\mathbb{R}^d} e^{-is\cdot(r\mp\nabla\omega_\sigma(\theta)\tau/\varepsilon)}e^{\pm i(\tau/2)s\cdot H_\sigma(\theta)s}\mu(\theta,ds). \tag{C.16}$$

Proof. We first substitute formula (C.4) in (C.11) and change the order of the integration $\mu(\theta,ds)\,d\varphi\to d\varphi\,\mu(\theta,ds)$. Therefore,

$$N^{\mp}_{\varepsilon,\sigma}(\tau,r;\theta) = (2\pi)^{-d}\int\limits_{\mathbb{R}^d} e^{-is\cdot r}\Big(\frac{1}{(4\pi k)^{d/2}}\int\limits_{[-\pi/\varepsilon,\pi/\varepsilon]^d} e^{-|s-\varphi|^2/(4k)}e^{\pm i\omega_\sigma(\theta+\varepsilon\varphi)\tau/\varepsilon^2}d\varphi\Big)\mu(\theta,ds). \tag{C.17}$$

Secondly, we replace $\omega_\sigma(\theta+\varepsilon\varphi)$ by $\omega_\sigma(\theta+\varepsilon s)$ in the inner integral. To do this we devide the integration into two: $|s-\varphi|\ge\varepsilon^\gamma$ and $|s-\varphi|\le\varepsilon^\gamma$ with a γ , $\gamma\in(1,\beta/2)$. For $|s-\varphi|\ge\varepsilon^\gamma$,

$$\frac{1}{(4\pi k)^{d/2}}\int\limits_{[-\pi/\varepsilon,\pi/\varepsilon]^d,|s-\varphi|>\varepsilon^\gamma} e^{-|s-\varphi|^2/(4k)}\,d\varphi \le \frac{e^{-\varepsilon^{2\gamma}/(4k)}}{(4\pi k)^{d/2}}\Big(\frac{\pi}{\varepsilon}\Big)^d \le C\,\frac{e^{-\varepsilon^{(2\gamma-\beta)}/4}}{\varepsilon^{\beta d/2+d}}\to 0 \tag{C.18}$$

as $\varepsilon\to+0$, since $k=\varepsilon^\beta$ and $\gamma<\beta/2$. Note that

$$\frac{1}{(4\pi k)^{d/2}}\int\limits_{[-\pi/\varepsilon,\pi/\varepsilon]^d} e^{-|s-\varphi|^2/(4k)}\,d\varphi = 1+o(1),\quad \varepsilon\to 0,$$

where $k=\varepsilon^\beta$. Therefore, for $|s-\varphi|\le\varepsilon^\gamma$, we estimate the difference

$$\frac{1}{(4\pi k)^{d/2}}\Big|\int\limits_{[-\pi/\varepsilon,\pi/\varepsilon]^d,|s-\varphi|<\varepsilon^\gamma} e^{-|s-\varphi|^2/(4k)}(e^{\pm i\omega_\sigma(\theta+\varepsilon\varphi)\tau/\varepsilon^2} - e^{\pm i\omega_\sigma(\theta+\varepsilon s)\tau/\varepsilon^2})\,d\varphi\Big|$$
$$\le \frac{C}{(4\pi k)^{d/2}}\int\limits_{[-\pi/\varepsilon,\pi/\varepsilon]^d} e^{-|s-\varphi|^2/(4k)}\varepsilon^{\gamma+1}\tau/\varepsilon^2\,d\varphi \le C\varepsilon^{\gamma-1}(1+o(1)), \tag{C.19}$$

where the last expression vanishes as $\varepsilon \to 0$ since $\gamma > 1$. By (C.18) and (C.19), the inner integral in (C.17) is

$$
\begin{aligned}
& \frac{1}{(4\pi k)^{d/2}} \int\limits_{[-\pi/\varepsilon,\pi/\varepsilon]^d, |s-\varphi|<\varepsilon^\gamma} e^{-|s-\varphi|^2/(4k)} e^{\pm i\omega_\sigma(\theta+\varepsilon\varphi)\tau/\varepsilon^2}\, d\varphi + o(1) \\
= \;& e^{\pm i\omega_\sigma(\theta+\varepsilon s)\tau/\varepsilon^2} \frac{1}{(4\pi k)^{d/2}} \int\limits_{[-\pi/\varepsilon,\pi/\varepsilon]^d, |s-\varphi|<\varepsilon^\gamma} e^{-|s-\varphi|^2/(4k)}\, d\varphi + o(1) \\
= \;& e^{\pm i\omega_\sigma(\theta+\varepsilon s)\tau/\varepsilon^2} \frac{1}{(4\pi k)^{d/2}} \int\limits_{[-\pi/\varepsilon,\pi/\varepsilon]^d} e^{-|s-\varphi|^2/(4k)}\, d\varphi + o(1) \\
= \;& e^{\pm i\omega_\sigma(\theta+\varepsilon s)\tau/\varepsilon^2} (1 + o(1)), \quad \varepsilon \to 0,
\end{aligned}
$$

uniformly in $\theta \in \mathbb{T}^d$. We substitute the last expression in (C.17) and get

$$
N^{\mp}_{\varepsilon,\sigma}(\tau, r; \theta) = (2\pi)^{-d} \int\limits_{\mathbb{R}^d} e^{-is\cdot r} e^{\pm i\omega_\sigma(\theta+\varepsilon s)\tau/\varepsilon^2}\, \mu(\theta, ds) + o(1), \quad \varepsilon \to 0,
$$

uniformly in $\theta \in \mathbb{T}^d$. Finally, we apply the Taylor sum representation (6.23) to $\omega_\sigma(\theta + \varepsilon s)$ and obtain (C.15)–(C.16). The integral in (C.16) can be taken over $|s| \le N$ with some $N \in \mathbb{N}$, by condition (2.14).

Step (v): We substitute $I_\varepsilon(\theta)$ of the form (C.10) with $N^{\mp}_{\varepsilon,\sigma}(\tau, r; \theta)$ from (C.13) (or (C.15)) in the r.h.s. of (6.17). Applying the decomposition (6.22) to $\hat{\mathcal{G}}^g_{\tau/\varepsilon^2}(\theta)^*$ we obtain (6.26), with $B^{\mp}_{\varepsilon,\sigma}$ (or $C^{\mp}_{\varepsilon,\sigma}$, respectively) instead of $A^{\mp}_{\varepsilon,\sigma}$. It remains to study the behaviour (as $\varepsilon \to 0$) of integrals of the form (6.27) with $B^{-}_{\varepsilon,\sigma}$ (or $C^{-}_{\varepsilon,\sigma}$, resp.) instead of $A^{-}_{\varepsilon,\sigma}$.

Let $\omega_\sigma(\theta) \pm \omega_{\sigma'}(\theta) \not\equiv \mathrm{const}_\pm$. In this case, the oscillatory integrals

$$
I^{\pm}_{\sigma\sigma'}(\varepsilon) \equiv (2\pi)^{-d} \frac{1}{2} \int\limits_{\mathbb{T}^d} e^{-i(l-p)\cdot\theta} e^{i(\omega_\sigma(\theta)\pm\omega_{\sigma'}(\theta))\tau/\varepsilon^2} h^{-}_\sigma(\theta) C^{-}_{\varepsilon,\sigma}(\tau, r; \theta) h^{\mp}_{\sigma'}(\theta)^*\, d\theta \qquad \text{(C.20)}
$$

vanish as $\varepsilon \to 0$, since $\sup\limits_{\varepsilon>0,\theta\in\mathbb{T}^d} |C^{-}_{\varepsilon,\sigma}(\tau, r; \theta)| \le C < \infty$, and hence, $h^{-}_\sigma(\theta) C^{-}_{\varepsilon,\sigma}(\tau, r; \theta) h^{\mp}_{\sigma'}(\theta)^* \in L^1(\mathbb{T}^d)$. The identities $\omega_\sigma(\theta) \pm \omega_{\sigma'}(\theta) \equiv \mathrm{const}_\pm$ in the exponent of (C.20) with $\mathrm{const}_\pm \ne 0$ are impossible by condition **E5**. The identity $\omega_\sigma(\theta) + \omega_{\sigma'}(\theta) \equiv 0$ implies $\omega_\sigma(\theta) \equiv \omega_{\sigma'}(\theta) \equiv 0$ which is impossible by **E4**. Therefore, if $\sigma \ne \sigma'$, $I^{\pm}_{\sigma\sigma'}(\varepsilon) = o(1)$ as $\varepsilon \to 0$. If $\sigma = \sigma'$, $I^{+}_{\sigma\sigma}(\varepsilon) = o(1)$, and only $I^{-}_{\sigma\sigma}(\varepsilon)$ contrubutes in the limit. Hence, (cf (6.28))

$$
S^{+}_{\varepsilon,\tau/\varepsilon^2} = (2\pi)^{-d} \frac{1}{2} \int\limits_{\mathbb{T}^d} e^{-i(l-p)\cdot\theta} \sum_{\sigma,\pm} h^{\mp}_\sigma(\theta) B^{\mp}_{\varepsilon,\sigma}(\tau, r; \theta) h^{\pm}_\sigma(\theta)^*\, d\theta + o(1), \quad \varepsilon \to 0. \qquad \text{(C.21)}
$$

Step (vi): To obtain (6.28), it remains to replace $B^{\mp}_{\varepsilon,\sigma}$ by $A^{\mp}_{\varepsilon,\sigma}$ in (C.21) (or $\hat{\mathbf{R}}_{r,\varepsilon}$ by $\hat{\mathbf{R}}_0$ in (C.14)). We check that the difference $B^{-}_{\varepsilon,\sigma} - A^{-}_{\varepsilon,\sigma}$ vanishes as $\varepsilon \to 0$ (for the difference $B^{+}_{\varepsilon,\sigma} - A^{+}_{\varepsilon,\sigma}$ the proof is similar). Indeed, by (3.13), (C.1), (C.14), and (6.25),

$$
A^{-}_{\varepsilon,\sigma} - B^{-}_{\varepsilon,\sigma} = C(\theta) \int\limits_{\mathbb{R}^d} \hat{\mathbf{R}}_0(r - \nabla\omega_\sigma(\theta)\tau/\varepsilon - x, \theta) \Big(1 - e^{-\varepsilon^\beta(\nabla\omega_\sigma(\theta)\tau/\varepsilon + x)^2}\Big) e^{-i/(2\tau)x\cdot H^{-1}_\sigma(\theta)x} dx,
$$

with $C(\theta) = e^{i\pi s/4}(2\pi\tau)^{-d/2}|\det H_\sigma(\theta)|^{-1/2}$. Write $c = 2|\tau| \max_{\theta\in\mathbb{T}^d,\sigma} |\nabla\omega_\sigma(\theta)|$. We devide the integration into two: $|x+\nabla\omega_\sigma(\theta)\tau/\varepsilon| \le c/\varepsilon$ and $|x+\nabla\omega_\sigma(\theta)\tau/\varepsilon| \ge c/\varepsilon$. For $|x+\nabla\omega_\sigma(\theta)\tau/\varepsilon| \le c/\varepsilon$,

$$\begin{aligned}
&\Big|\int\limits_{|x+\nabla\omega_\sigma(\theta)\tau/\varepsilon|\le c/\varepsilon} \hat{\mathbf{R}}_0(r-\nabla\omega_\sigma(\theta)\tau/\varepsilon - x,\theta)\Big(1-e^{-\varepsilon^\beta(\nabla\omega_\sigma(\theta)\tau/\varepsilon+x)^2}\Big)e^{-i/(2\tau)x\cdot H_\sigma^{-1}(\theta)x}\,dx\Big| \\
\le\ & C \int\limits_{|x+\nabla\omega_\sigma(\theta)\tau/\varepsilon|\le c/\varepsilon} \varepsilon^\beta|x+\nabla\omega_\sigma(\theta)\tau/\varepsilon|^2\,dx \le C_1\varepsilon^{\beta-2-d}\to 0,\quad \varepsilon\to 0,
\end{aligned}$$

since $\beta > d+2$. For $|x+\nabla\omega_\sigma(\theta)\tau/\varepsilon| \ge c/\varepsilon$, we apply the integration by parts in every variable x_i, $i=1,\ldots,d$, For simplicity, let $d=1$ and h stand for $-1/(2\tau)H_\sigma^{-1}(\theta)$. Therefore,

$$\begin{aligned}
&\int\limits_{c/\varepsilon-\omega'_\sigma(\theta)\tau/\varepsilon}^{+\infty} \hat{\mathbf{R}}_0(r-\omega'_\sigma(\theta)\tau/\varepsilon - x,\theta)\Big(1-e^{-\varepsilon^\beta(\omega'_\sigma(\theta)\tau/\varepsilon+x)^2}\Big)e^{ihx^2}\,dx \\
=\ & \frac{1}{2ih}\Big[-\hat{\mathbf{R}}_0(r-c/\varepsilon,\theta)\Big(1-e^{-\varepsilon^\beta(c/\varepsilon)^2}\Big)\frac{e^{ihx^2}}{x}\Big|_{x=(c-\omega'_\sigma(\theta)\tau)/\varepsilon} \\
&+\int\limits_{(c-\omega'_\sigma(\theta)\tau)/\varepsilon}^{+\infty} \partial_r\hat{\mathbf{R}}_0(r-\omega'_\sigma(\theta)\tau/\varepsilon - x,\theta)\Big(1-e^{-\varepsilon^\beta(\omega'_\sigma(\theta)\tau/\varepsilon+x)^2}\Big)\frac{e^{ihx^2}}{x}\,dx \\
&-\varepsilon^\beta 2\int\limits_{(c-\omega'_\sigma(\theta)\tau)/\varepsilon}^{+\infty} \hat{\mathbf{R}}_0(r-\omega'_\sigma(\theta)\tau/\varepsilon - x,\theta)(\omega'_\sigma(\theta)\tau/\varepsilon+x)e^{-\varepsilon^\beta(\omega'_\sigma(\theta)\tau/\varepsilon+x)^2}\frac{e^{ihx^2}}{x}\,dx \\
&+\int\limits_{(c-\omega'_\sigma(\theta)\tau)/\varepsilon}^{+\infty} \hat{\mathbf{R}}_0(r-\omega'_\sigma(\theta)\tau/\varepsilon - x,\theta)e^{-\varepsilon^\beta(\omega'_\sigma(\theta)\tau/\varepsilon+x)^2}\frac{e^{ihx^2}}{x^2}\,dx\Big] \\
=\ & I_1+I_2+I_3+I_4.
\end{aligned}$$

The terms I_1 and I_4 tend to zero as $\varepsilon\to 0$, since $|\hat{\mathbf{R}}_0| \le C < \infty$ by condition **I1**. Note that $|x| \ge c/(2\varepsilon)$ if $|x+\omega'_\sigma(\theta)\tau/\varepsilon| \ge c/\varepsilon$. Hence,

$$\begin{aligned}
|I_3| &\le C_1\varepsilon^\beta \int\limits_{(c-\omega'_\sigma(\theta)\tau)/\varepsilon}^{+\infty} (\omega'_\sigma(\theta)\tau/\varepsilon+x)e^{-\varepsilon^\beta(\omega'_\sigma(\theta)\tau/\varepsilon+x)^2}\frac{1}{|x|}dx \\
&\le C_2\varepsilon^{\beta+1}\int\limits_{c/\varepsilon}^{+\infty} ye^{-\varepsilon^\beta y^2}dy = C_3\,\varepsilon\, e^{-\varepsilon^\beta(c/\varepsilon)^2}\to 0,\quad \varepsilon\to 0.
\end{aligned}$$

To prove that the contribution of the integral I_2 vanishes, we repeat the integration by parts and use the bound $\sup_{r\in\mathbb{R}^d,\theta\in\mathbb{T}^d} |\partial_r^k\hat{\mathbf{R}}_0(r,\theta)| \le C < \infty$, where $k\in[0, d^2/2+2d]$. This bound follows from condition (2.13)–(2.14).

Appendix D: Local conservation law

Let $v(x,t)$ be a solution of (2.1) with the finite energy. The local energy in the point $x\in\mathbb{Z}^d$ is defined as

$$\mathcal{E}(x,t):=\frac{1}{2}\Big\{|\dot v(x,t)|^2+\sum_{y\in\mathbb{Z}^d}v(x,t)\cdot V(x-y)v(y,t)\Big\}.$$

We derive formally the expression for the energy current of the finite energy solutions $v(x,t)$. For the half-space $\Omega_k:=\{x\in\mathbb{Z}^d:\ x=(x_1,\dots,x_d),\ x_k\le 0\}$, $k=1,\dots,d$, we define the energy in the region Ω_k as

$$\mathcal{E}_{\Omega_k}(t):=\frac{1}{2}\sum_{x\in\Omega_k}\Big\{|\dot v(x,t)|^2+\sum_{y\in\mathbb{Z}^d}v(x,t)\cdot V(x-y)v(y,t)\Big\}.$$

By formal calculation, using Eqn (2.1) we obtain $\dot{\mathcal{E}}_{\Omega_k}(t)=-\sum\limits_{x'\in\mathbb{Z}^d:\,x'_k=0}j_k(x',t)$, where $j_k(x',t)$ stands for the energy current density in the direction $e_k=(\delta_{1k},\dots,\delta_{dk})$, $k=1,\dots,d$, $x'\in\mathbb{Z}^d$ with $x'_k=0$,

$$\begin{aligned}j_k(x',t)\ &:=\ -\frac{1}{2}\sum_{y'\in\mathbb{Z}^d:\,y'_k=0}\Big\{\sum_{m\ge1,\,p\le0}\dot v(x'+me_k,t)\cdot V(x'+me_k-y'-pe_k)v(y'+pe_k,t)\\ &\qquad-\sum_{m\le0,\,p\ge1}\dot v(x'+me_k,t)\cdot V(x'+me_k-y'-pe_k)v(y'+pe_k,t)\Big\}.\end{aligned}\tag{D.1}$$

Now let $v(x,t)$ be the random solution to (2.1) with the initial measure μ_0^ε satisfying **V1** and **V2**. Therefore, for any $x\in\mathbb{Z}^d$, $\tau\in\mathbb{R}\setminus0$, and $r\in\mathbb{R}^d$, the *average energy* is

$$\begin{aligned}&\mathbb{E}_0^\varepsilon[\mathcal{E}(x+[r/\varepsilon],\tau/\varepsilon)]=\mathbb{E}_{\tau/\varepsilon}^\varepsilon[\mathcal{E}(x+[r/\varepsilon],0)]\\ &=\frac{1}{2}\operatorname{tr}\Big[Q^{11}_{\varepsilon,\tau/\varepsilon}([r/\varepsilon]+x,[r/\varepsilon]+x)+\sum_{y\in\mathbb{Z}^d}Q^{00}_{\varepsilon,\tau/\varepsilon}([r/\varepsilon]+x,[r/\varepsilon]+y)\cdot V^T(x-y)\Big],\end{aligned}$$

by condition **E1** and the uniform bound for the correlation functions of the measure $\mu^\varepsilon_{\tau/\varepsilon}$ (see [14, Lemma 5.1] or Lemma 7.1):

$$\sup_{\varepsilon>0}\sup_{i,j=0,1}\sup_{z,z'\in\mathbb{Z}^d}\|Q^{ij}_{\varepsilon,\tau/\varepsilon}(z,z')\|\le C<\infty.\tag{D.2}$$

It follows from Theorem 3.2 that for any $\tau\ne0$, $r\in\mathbb{R}^d$, $\mathbb{E}_0^\varepsilon[\mathcal{E}(x+[r/\varepsilon],\tau/\varepsilon)]\to\mathbf{e}(\tau,r)$ as $\varepsilon\to0$, where

$$\begin{aligned}\mathbf{e}(\tau,r)\ &=\ \frac{1}{2}\operatorname{tr}\Big[q^{11}_{\tau,r}(0)+\sum_{x\in\mathbb{Z}^d}q^{00}_{\tau,r}(x)V^T(x)\Big]\\ &=\ \frac{1}{2}(2\pi)^{-d}\operatorname{tr}\int_{\mathbb{T}^d}\Big(\hat q^{11}_{\tau,r}(\theta)+\hat q^{00}_{\tau,r}(\theta)\hat V^*(\theta)\Big)\,d\theta=(2\pi)^{-d}\operatorname{tr}\int_{\mathbb{T}^d}\hat q^{11}_{\tau,r}(\theta)\,d\theta.\end{aligned}\tag{D.3}$$

The last equality follows from condition **E2** and Remarks 3.4 (i).

Similarly, by (D.1) we obtain

$$\mathbb{E}_0^\varepsilon[j_k(x',t)] = \frac{1}{2}\sum_{y'\in\mathbb{Z}^d:\, y'_k=0}\Big(\sum_{m\le 0,\, p\ge 1}\mathrm{tr}\Big[Q^{10}_{\varepsilon,t}(x'+me_k,y'+pe_k)\cdot V^T(x'-y'+(m-p)e_k)\Big]$$
$$-\sum_{m\ge 1,\, p\le 0}\mathrm{tr}\Big[Q^{10}_{\varepsilon,t}(x'+me_k,y'+pe_k)\cdot V^T(x'-y'+(m-p)e_k)\Big]\Big).$$

Therefore, by Theorem 3.2, the following limit exists, $\lim\limits_{\varepsilon\to 0}\mathbb{E}_0^\varepsilon[j_k(x+[r/\varepsilon],\tau/\varepsilon)] = \mathbf{j}_k(\tau,r)$ for any $\tau\ne 0$, $r\in\mathbb{R}^d$, $k=1,\dots,d$. Here

$$\mathbf{j}_k(\tau,r) = \frac{1}{2}\sum_{y'\in\mathbb{Z}^d:\, y'_k=0}\Big(\sum_{m\le -1,\, p\ge 0}\mathrm{tr}\Big[q^{10}_{\tau,r}(x'-y'+(m-p)e_k)V^T(x'-y'+(m-p)e_k)\Big]$$
$$-\sum_{m\ge 0,\, p\le -1}\mathrm{tr}\Big[q^{10}_{\tau,r}(x'-y'+(m-p)e_k)V^T(x'-y'+(m-p)e_k)\Big]\Big).$$

Write $x'-y' =: z'\in\mathbb{Z}^d$ with $z'_k=0$, $m-p := s\in\mathbb{Z}^1$ and change the order of the summation in the series. Therefore,

$$\mathbf{j}_k(\tau,r) = -\frac{1}{2}\sum_{z'\in\mathbb{Z}^d:\, z'_k=0}\sum_{s\in\mathbb{Z}^1}\mathrm{tr}[q^{10}_{\tau,r}(z'+se_k)V^T(z'+se_k)]s = -\frac{1}{2}\sum_{z\in\mathbb{Z}^d}\mathrm{tr}[q^{10}_{\tau,r}(z)z_kV^T(z)]$$
$$= -\frac{i}{2}(2\pi)^{-d}\,\mathrm{tr}\int_{\mathbb{T}^d}\hat{q}^{10}_{\tau,r}(\theta)\partial_k\hat{V}(\theta)\,d\theta,\quad k=1,\dots,d. \tag{D.4}$$

Write $\mathbf{j}(\tau,r) = (\mathbf{j}_1(\tau,r),\dots,\mathbf{j}_d(\tau,r))$. Finally, (D.3), (D.4) and Corollary 3.3 yield

$$\partial_\tau\mathbf{e}(\tau,r) + \nabla_r\cdot\mathbf{j}(\tau,r) = 0,\quad \tau\in\mathbb{R},\quad r\in\mathbb{R}^d.$$

Remark. In Section 2.3, we give the example of the "*local equilibrium*" initial measures μ_0^ε. Namely, let $q_0^{ij}(z)$ be the correlation functions of the Gibbs measure g (see Definition 2.5) with $\beta=1$, i.e., $\hat{q}_0^{00}(\theta) = \hat{V}^{-1}(\theta)$, $\hat{q}_0^{11}(\theta) = I$, $\hat{q}_0^{01}(\theta) = \hat{q}_0^{10}(\theta) = 0$. Put $\hat{\mathbf{R}}_0(r,\theta) = T(r)\hat{q}_0(\theta)$, where the function $T(r)$, $r\in\mathbb{R}^d$, is defined in Section 2.3. Moreover, let μ_0^ε, $\varepsilon>0$, be Gaussian measures with the correlation functions $Q_\varepsilon^{ij}(z,z')$ defined in (2.17). Then conditions **V1** and **V2** hold. In this case, the limit correlation matrices $\hat{q}^{ij}_{\tau,r}(\theta)$ have a form

$$\hat{q}^{11}_{\tau,r}(\theta) = \hat{V}(\theta)\hat{q}^{00}_{\tau,r}(\theta) = \sum_{\sigma=1}^s\mathbf{T}^\sigma_+(\tau,r;\theta)\Pi_\sigma(\theta),$$
$$\hat{q}^{01}_{\tau,r}(\theta) = -\hat{q}^{10}_{\tau,r}(\theta) = i\sum_{\sigma=1}^s\mathbf{T}^\sigma_-(\tau,r;\theta)\omega_\sigma^{-1}(\theta)\Pi_\sigma(\theta),$$

where, by definition, $\mathbf{T}^\sigma_\pm(\tau,r;\theta) := (1/2)(T(r+\nabla\omega_\sigma(\theta)\tau)\pm T(r-\nabla\omega_\sigma(\theta)\tau))$. Therefore,

$$\mathbf{e}(\tau,r) = (2\pi)^{-d}\sum_{\sigma=1}^s\int_{\mathbb{T}^d}\mathbf{T}^\sigma_+(\tau,r;\theta)\,\mathrm{tr}\,\Pi_\sigma(\theta)\,d\theta,$$
$$\mathbf{j}(\tau,r) = -(2\pi)^{-d}\sum_{\sigma=1}^s\int_{\mathbb{T}^d}\mathbf{T}^\sigma_-(\tau,r;\theta)\nabla\omega_\sigma(\theta)\,\mathrm{tr}\,\Pi_\sigma(\theta)\,d\theta.$$

In [6], the "*locally conserved*" quantities were studied in the case when $d = 1$. Let us consider these quantities in many-dimensional case. At first, introduce the following matrix-valued functions

$$
\begin{aligned}
E(z+h,z;X_0) &:= \frac{1}{2}\Big(v_1(z+h)\otimes v_1(z) + v_0(z+h)\otimes \sum_{z'\in\mathbb{Z}^d} V(z-z')v_0(z')\Big),\\
A(z+h,z;X_0) &:= \frac{1}{2}\Big(v_0(z+h)\otimes v_1(z) - v_1(z+h)\otimes v_0(z)\Big), \quad z,h\in\mathbb{Z}^d, \quad X_0=(v_0,v_1).
\end{aligned}
$$

The "locally conserved" quantities $X_h^\varepsilon(\varphi,X_0)$, $Y_h^\varepsilon(\varphi,X_0)$ are defined as follows. For $\varphi \in C_0^1(\mathbb{R}^d)$, we set

$$
\begin{aligned}
X_h^\varepsilon(\varphi,X_0) &:= \varepsilon^d \sum_{z\in\mathbb{Z}^d} \varphi(\varepsilon z) E(z+h,z;X_0),\\
Y_h^\varepsilon(\varphi,X_0) &:= \varepsilon^d \sum_{z\in\mathbb{Z}^d} \varphi(\varepsilon z) A(z+h,z;X_0), \quad h\in\mathbb{Z}^d, \quad \varepsilon>0.
\end{aligned}
$$

Theorem D.1 *Let conditions* **I1**–**I4**, **V1**, **V2** *hold. Then for any* $\varphi \in C_0^1(\mathbb{R}^d)$, $h\in\mathbb{Z}^d$, $\tau \neq 0$*, there exist the limits*

$$
\lim_{\varepsilon\to 0} \mathbb{E}^\varepsilon_{\tau/\varepsilon}[X_h^\varepsilon(\varphi,\cdot)] = \mathbf{E}(\varphi;\tau,h), \quad \lim_{\varepsilon\to 0} \mathbb{E}^\varepsilon_{\tau/\varepsilon}[Y_h^\varepsilon(\varphi,\cdot)] = \mathbf{A}(\varphi;\tau,h).
$$

The matrices $\mathbf{E}(\varphi;\tau,h)$ *and* $\mathbf{A}(\varphi;\tau,h)$ *are given by their Fourier transform in the following way. Write*

$$
\hat{\mathbf{E}}(\varphi;\tau,\theta) := \sum_{h\in\mathbb{Z}^d} e^{ih\cdot\theta}\mathbf{E}(\varphi;\tau,h), \quad \hat{\mathbf{A}}(\varphi;\tau,\theta) := \sum_{h\in\mathbb{Z}^d} e^{ih\cdot\theta}\mathbf{A}(\varphi;\tau,h), \quad \theta\in\mathbb{T}^d.
$$

Then

$$
\hat{\mathbf{E}}(\varphi;\tau,\theta) = \frac{1}{2}\int_{\mathbb{R}^d} \varphi(r)\Big(\hat{q}^{11}_{\tau,r}(\theta) + \hat{q}^{00}_{\tau,r}(\theta)\hat{V}(\theta)\Big)\,dr = \int_{\mathbb{R}^d} \varphi(r)\hat{q}^{11}_{\tau,r}(\theta)\,dr, \tag{D.5}
$$

$$
\hat{\mathbf{A}}(\varphi;\tau,\theta) = \frac{1}{2}\int_{\mathbb{R}^d} \varphi(r)\Big(\hat{q}^{01}_{\tau,r}(\theta) - \hat{q}^{10}_{\tau,r}(\theta)\Big)\,dr = \int_{\mathbb{R}^d} \varphi(r)\hat{q}^{01}_{\tau,r}(\theta)\,dr, \tag{D.6}
$$

where $\hat{q}^{ij}_{\tau,r}(\theta)$ *are defined in (3.1).*

The proof of Theorem D.1 is based on Theorem 3.2 and the bound (D.2).

Remark It follows from (D.5), (D.6) and Corollary 3.3 that $\hat{\mathbf{E}}$ and $\hat{\mathbf{A}}$ satisfy the equations

$$
\partial_\tau \hat{\mathbf{E}}(\varphi;\tau,\theta) = i\omega_\sigma(\theta)\int_{\mathbb{R}^d} \nabla\omega_\sigma(\theta)\cdot\nabla\varphi(r)\hat{q}^{01}_{\tau,r}(\theta)\,dr = i\omega_\sigma(\theta)\nabla\omega_\sigma(\theta)\cdot\hat{\mathbf{A}}(\nabla\varphi;\tau,\theta),
$$

$$
\partial_\tau \hat{\mathbf{A}}(\varphi;\tau,\theta) = -i\omega_\sigma^{-1}(\theta)\int_{\mathbb{R}^d} \nabla\omega_\sigma(\theta)\cdot\nabla\varphi(r)\hat{q}^{11}_{\tau,r}(\theta)\,dr = -i\omega_\sigma^{-1}(\theta)\nabla\omega_\sigma(\theta)\cdot\hat{\mathbf{E}}(\nabla\varphi;\tau,\theta).
$$

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